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Index Formulas for Generalized Wiener–Hopf Operators and Boson–Fermion Correspondence in 2*N* Dimensions

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Abstract. The kernels of operators associated with special chiral gauge transformations ('kinks') in the $2N$-dimensional Dirac theory are explicitly determined. The result is used to obtain index formulas for Fredholm operators corresponding to continuous chiral gauge transformations. Moreover, the Fock space quadratic forms corresponding to the kinks are proved to converge to the Dirac field as the kink size goes to zero. It is also shown that for $N \equiv 1, 2 (\mathrm{mod}\, 4)$ the Majorana field can be reached in a similar fashion.

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1. Introduction

The inspiration for this paper came from previous work of Matsui [1] and from joint work with Carey [2]. Some of its results have been announced in [3]. In [1] Matsui proves an Atiyah–Singer type index formula for scattering operators arising for certain time-dependent external fields in the context of the four-dimensional single particle Dirac equation. He also obtains index formulas for time-independent unitary gauge transformations. His main tools are results from the theory of pseudo-differential operators and an index theorem due to Hörmander. Recently, he has lifted some technical restrictions and obtained extensions to $2N$ dimensions [4].

Inasmuch as the results of this paper overlap with those of Matsui, they are arrived at in a quite different fashion. This will be clear from the sketch of the paper which now follows. After setting up notation in Subsect. 2.1, we present our key result, Theorem 2.1 in Subsect. 2.2. In this theorem the kernels of certain operators are determined explicitly. These operators are associated with special chiral gauge transformations (dubbed standard kinks) in the framework of the $2N$-dimensional single particle Dirac theory. The kernel determination has algebraic aspects that are dealt with in Appendix A, cf. Lemmas A1, A2, and analytic aspects that are handled in Appendix B, cf. Lemma B1. The proofs of the former lemmas are self-contained, whereas the proof of the latter lemma makes use of results from the theory of Schrödinger operators, all of which can be found in [5–8].

In Subsect. 2.3 properties of bounded matrix-valued multiplication operators are derived (Theorems 2.3–2.7) by using results on compactness and non-compactness of operators having Schwartz kernels with certain properties. The latter results are largely self-contained and can be found in Appendix C. Subsequently, unitary multipliers are studied in Subsect. 2.4. Using the explicit information on the standard kinks and Bott periodicity, index formulas for continuous chiral gauge transformations with constant asymptotics and with "hedge-hog" asymptotics for $|x| \to \infty$ are proved in Theorems 2.8 and 2.9, respectively.

For $N = 1$ and vanishing particle mass the multipliers studied in Sect. 2 give rise to (matrix-valued) Wiener–Hopf operators. The kernel problem for the standard kinks is trivial in this case, since one is in essence dealing with one-sided shifts. However, the fact that the relevant kernels can be found explicitly for $N = 1$ and $m > 0$ is already quite non-obvious and surprising. This state of affairs was first pointed out and exploited in [2] to study the gauge groups arising in the massive second-quantized Dirac theory in 2D via a rigorous version of boson–fermion correspondence. Specifically, in [2] the $N = 1$ standard kinks are proved to generate Bogoliubov transformations whose renormalized unitary implementers converge to the free Dirac field as a scale parameter describing the kink size goes to 0. The connection of this result to boson–fermion correspondence is discussed in [2], and a corresponding "abstract" picture is sketched in [3].

In Sect. 3 we present the generalization of this convergence result and its "neutral analog" to the arbitrary N case, cf. Theorem 3.1 in Subsect. 3.2 and

Theorems 3.2 and 3.3 in Subsect. 3.3. This entails a change in perspective detailed in Subsect. 3.1, and some information on charge conjugation assembled in Appendix A. Moreover, a crucial technical result is relegated to Appendix D. We should mention that the mathematical context of the results in Sect. 3 is possibly not sufficiently explained in this paper; for more background the reader might consult [2, 3, 9] and references given there.

The paper is concluded with Appendix E, where the main text is linked up with the external field problem in the Dirac theory. The results obtained there should be compared with the external field index formulas obtained by Matsui [1]. Further work concerning index theorems on open manifolds includes the recent paper [10], which also lists other references in this area.

Throughout the paper $2N$ denotes the space-time dimension, whereas the symbol n is reserved for the integer 2^{N-1}.

2. Matrix Multipliers in the One-Particle Dirac Theory

2.1. Preliminaries. In this subsection we introduce operators arising in the Dirac description of a particle in a $2N$-dimensional Minkowski space-time. (Several more such operators, which are not needed till Sect. 3, will be introduced in Subsect. 3.1.) First of all, the Dirac Hamiltonian $\check{H}$ is the operator on $L^2(\mathbb{R}^{2N-1}, dx)^{2n}$ with domain the Sobolev space $H_1(\mathbb{R}^{2N-1})^{2n}$, whose action is given by

$$\begin{pmatrix} -i\sigma\cdot\nabla & m1_n \\ m1_n & i\sigma\cdot\nabla \end{pmatrix}, \quad m \geqq 0. \tag{2.1}$$

Here and below, differentiations act weakly. Also, $\sigma_1, \ldots, \sigma_{2N-1}$ denote self-adjoint $n \times n$ matrices representing the Euclidean Clifford algebra in $\mathbb{R}^{2N-1}$, and the decomposition of $\mathbb{C}^{2n} \cong \mathscr{F}_a(\mathbb{C}^N)$ used in (2.1) is explained in Appendix A. Clearly, $\check{H}$ is a self-adjoint operator.

We shall employ Fourier transformation

$$\mathscr{F}: \check{\mathscr{H}} \equiv L^2(\mathbb{R}^{2N-1}, dx)^{2n} \to \mathscr{H} \equiv L^2(\mathbb{R}^{2N-1}, dp)^{2n},$$
$$(\mathscr{F}f)(p) \equiv (2\pi)^{-(2N-1)/2} \int dx \exp(-ip\cdot x) f(x) \tag{2.2}$$

to transform operators $\check{A}$ on $\check{\mathscr{H}}$ to operators A on $\mathscr{H}$ and vice versa, i.e.,

$$A \equiv \mathscr{F} \check{A} \mathscr{F}^{-1}. \tag{2.3}$$

With this convention we obtain

$$H = \begin{pmatrix} \sigma\cdot p & m1_n \\ m1_n & -\sigma\cdot p \end{pmatrix} = \alpha\cdot p + \beta m. \tag{2.4}$$

Hence,

$$H^2 = E_p^2 1_{2n}, \quad E_p \equiv (p^2 + m^2)^{1/2}, \tag{2.5}$$

so the projections $P_\pm$ on the positive and negative spectral subspaces of H are given by the multipliers

$$P_\pm(p) = \tfrac{1}{2}(1 \pm H(p)/E_p). \tag{2.6}$$

We shall use the notation

$$\mathscr{H}_\delta \equiv P_\delta \mathscr{H}, \quad \delta = +, -, \tag{2.7}$$

$$A_{\delta\delta'} \equiv P_\delta A P_{\delta'}, \quad \delta, \delta' = +, -, \tag{2.8}$$

where A is an operator on $\mathscr{H}$. Note that

$$A_{\delta\delta'}{}^* = A^*{}_{\delta'\delta}. \tag{2.9}$$

Next, we introduce the parity operator

$$(\check{P}g)(x) \equiv \beta f(-x), \tag{2.10}$$

and the scaling group

$$(\check{D}(\varepsilon)f)(x) \equiv \varepsilon^{(2N-1)/2} f(\varepsilon x), \quad \varepsilon \in (0, \infty). \tag{2.11}$$

Clearly, P and $D(\varepsilon)$ are unitary, and one has

$$[P, P_\delta] = 0, \tag{2.12}$$

whereas the relation

$$[D(\varepsilon), P_\delta] = 0, \quad m = 0 \tag{2.13}$$

does not hold for $m > 0$. Similarly, the chiral projections

$$\check{q}_+ \equiv \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}, \quad \check{q}_- \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix} \tag{2.14}$$

satisfy

$$[q_s, P_\delta] = 0, \quad s = +, -, \quad m = 0 \tag{2.15}$$

but do not commute with P_δ for $m > 0$.

From now on, we shall assume that an internal symmetry space $\mathbb{C}^k (k \geqq 1)$ is tensored on to $\check{\mathscr{H}}$ and $\mathscr{H}$, and we shall denote the resulting Hilbert spaces again by $\check{\mathscr{H}}$ and $\mathscr{H}$. Tensoring the above operators with 1_k yields operators that will be denoted by the same symbols, whenever no confusion is likely to arise. When this is done, all of the above formulas and relations are valid as they stand.

2.2. The Standard Kinks. In this subsection we study unitary matrix multiplication operators on $\check{\mathscr{H}}$ that reduce to the standard kinks of [2] for $N = 1$. First, we take $\mathbb{C}^k \equiv \mathbb{C}^n$ and set

$$\check{K}_{+,\varepsilon} \equiv \begin{bmatrix} 1_n \otimes \dfrac{\sigma \cdot x + (-)^N i\varepsilon 1_n}{\sigma \cdot x - (-)^N i\varepsilon 1_n} & 0 \\ 0 & 1_n \otimes 1_n \end{bmatrix}, \tag{2.16}$$

$$\check{K}_{-,\varepsilon} \equiv \begin{bmatrix} 1_n \otimes 1_n & 0 \\ 0 & 1_n \otimes \dfrac{\sigma \cdot x - (-)^N i\varepsilon 1_n}{\sigma \cdot x + (-)^N i\varepsilon 1_n} \end{bmatrix}, \tag{2.17}$$

where $\varepsilon \in (0, \infty)$. Note that one has

$$P K_{s,\varepsilon} P = K_{-s,\varepsilon}, \quad s = +, -, \tag{2.18}$$

$$K_{s,\varepsilon} = D(\varepsilon)^* K_{s,1} D(\varepsilon) \tag{2.19}$$

in view of (2.10) and (2.11). The importance of these multipliers (henceforth referred to as standard kinks) hinges on the following result.

Theorem 2.1. *The kernel of $K^*_{s,\varepsilon--}$ is trivial, whereas the kernel of $K_{s,\varepsilon--}$ is spanned by*

$$\kappa_{s,\varepsilon,-}(p) \equiv -\exp(-\varepsilon E_p)P_-(p)\begin{cases}\begin{pmatrix}u\\0\end{pmatrix}, & s=+\\[2mm] \begin{pmatrix}0\\u\end{pmatrix}, & s=-\end{cases}, \tag{2.20}$$

where $u \in \mathbb{C}^n \otimes \mathbb{C}^n$ is the unit vector of Appendix A. Moreover, the function

$$\kappa_{s,\varepsilon,+} \equiv K_{s,\varepsilon}\kappa_{s,\varepsilon,-} \tag{2.21}$$

is given by

$$\kappa_{s,\varepsilon,+}(p) = \exp(-\varepsilon E_p)P_+(p)\begin{cases}\begin{pmatrix}u\\0\end{pmatrix}, & s=+\\[2mm] \begin{pmatrix}0\\u\end{pmatrix}, & s=-\end{cases}. \tag{2.22}$$

Proof. Due to (2.18) and (2.12) we need only prove this for $s=+$. We shall from now on suppress the subscripts $+,\varepsilon$ to ease the notation. We begin by noting that the kernel of K_{--} consists of those vectors $\kappa_- \in \mathscr{H}_-$ for which $K\kappa_-$ belongs to $\mathscr{H}_+$. Also, since K is unitary, one has

$$\operatorname{Ker} K^*_{--} = K \operatorname{Ker} K_{++}. \tag{2.23}$$

Therefore, we shall study the equation

$$K\kappa_\delta = \kappa_{-\delta} \quad \kappa_\delta \in \mathscr{H}_\delta \tag{2.24$_\delta$}$$

and show that $(2.24)_+$ has no non-trivial solutions, and that any solution to $(2.24)_-$ is a multiple of $\kappa_{+,\varepsilon,-}(p)$. To this end we set

$$f \equiv \kappa_+ + \kappa_-, \quad g \equiv \kappa_+ - \kappa_- \tag{2.25}$$

so that

$$f = \frac{1}{E_p} Hg, \tag{2.26}$$

and rewrite $(2.24)_\delta$ as

$$\begin{pmatrix}1_n \otimes \sigma\cdot\nabla & 0\\ 0 & 1_n \otimes 1_n\end{pmatrix} g = (-)^{N-1}\delta\varepsilon \begin{pmatrix}1_n \otimes 1_n & 0\\ 0 & 0\end{pmatrix} f. \tag{2.27}$$

Hence one concludes that $(2.24)_\delta$ implies

$$g = \begin{pmatrix}G\\0\end{pmatrix}, \quad f = \frac{1}{E_p}\begin{pmatrix}(\sigma\cdot p \otimes 1_n)G\\ mG\end{pmatrix}, \tag{2.28}$$

where G satisfies

$$(1_n \otimes \sigma\cdot\nabla)G(p) = (-)^{N-1}\frac{\delta\varepsilon}{E_p}(\sigma\cdot p \otimes 1_n)G(p). \tag{2.29$_\delta$}$$

Conversely, if $G_\delta(p) \in L^2(\mathbb{R}^{2N-1}) \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ satisfies $(2.29)_\delta$, then g, f defined by (2.28) satisfy (2.26), (2.27). Also, setting $\kappa_\delta \equiv \frac{1}{2}(f + \delta g)$ one infers that $(2.24)_\delta$ holds true. Thus we need only study $(2.29)_\delta$. For $N = 1$ this is elementary: The solutions $c\exp(-\varepsilon E_p)$ to the first order ODE $(2.29)_-$ are in $L^2(\mathbb{R})$, whereas the solutions $c\exp(\varepsilon E_p)$ to $(2.29)_+$ are not. For $N > 1$ we invoke Lemma B1 to conclude that $(2.29)_+$ has no non-zero L^2-solutions, whereas any L^2-solution to $(2.29)_-$ is proportional to

$$G_-(p) \equiv \exp(-\varepsilon E_p)u. \tag{2.30}$$

This gives rise to the functions $\kappa_{+,\varepsilon,\delta}(p)$ in the way just explained. □

Next, we consider a second generalization of the $N = 1$ kinks of [2] which is more obvious, inasmuch as no internal symmetry space is needed. It consists in taking

$$\check{K}'_{+,\varepsilon} \equiv \begin{bmatrix} \dfrac{\sigma\cdot x - i\varepsilon 1_n}{\sigma\cdot x + i\varepsilon 1_n} & 0 \\ 0 & 1_n \end{bmatrix}, \quad K'_{-,\varepsilon} \equiv PK'_{+,\varepsilon}P. \tag{2.31}$$

However, this generalization is "wrong," as will be clear from what follows.

Theorem 2.2. *For $N > 1$ one has*

$$\dim \operatorname{Ker} K'^*_{s,\varepsilon--} = 0, \tag{2.32}$$

$$\dim \operatorname{Ker} K'_{s,\varepsilon--} = \infty. \tag{2.33}$$

Proof. Proceeding as in the previous case, we arrive at obvious analogs of (2.23)–$(2.29)_\delta$. In particular, the kernel problem can be reduced to finding the L^2-solutions of

$$\sigma\cdot\nabla G(p) = \frac{\delta\varepsilon}{E_p}\sigma\cdot pG(p). \tag{2.34$_\delta$}$$

Picking $\delta = +$ and setting $G \equiv \exp(\varepsilon E_p)H$ yields $\sigma\cdot\nabla H = 0$. But if G is L^2, then H is also L^2, so that $\sigma\cdot x\hat{H}(x) = 0$, with $\hat{H}$ the Fourier transform of H. Thus, we must have $H = 0$, so (2.32) follows.

Now consider $(2.34)_-$. Setting $G \equiv \exp(-\varepsilon E_p)H$, we get again $\sigma\cdot\nabla H = 0$. But if we now take H equal to one of the columns of the matrix $\sigma\cdot\nabla P$ with $P(p)$ an arbitrary harmonic polynomial, then G not only solves $(2.34)_-$, but is also L^2. Therefore, we may conclude that (2.33) holds true. □

2.3. Bounded Multipliers. In this subsection we consider bounded operators on

$$\check{\mathscr{H}} \equiv L^2(\mathbb{R}^{2N-1}, dx) \otimes \mathbb{C}^{2n} \otimes \mathbb{C}^k, \quad k \geqq 1 \tag{2.35}$$

of the form

$$(\check{M}f)(x) \equiv \mu(x)f(x) \quad f \in \check{\mathscr{H}}, \tag{2.36}$$

where μ is a $(2nk \times 2nk)$-matrix-valued function on $\mathbb{R}^{2N-1}$. Such operators form a W^*-algebra henceforth denoted $\check{\mathscr{G}}$. Our aim is to obtain conditions on μ

guaranteeing that the off-diagonal parts $M_{\delta,-\delta}$ are compact or not, and Hilbert–Schmidt (HS) or not. Clearly, this is equivalent to $[P_+,M]$ having this property or not. We shall study this problem by applying the results of Appendix C to the Schwartz kernel of $[P_+,M]$. This kernel is proportional to

$$C_M(p,q)\equiv\frac{1}{E_p}\begin{pmatrix}\sigma\cdot p & m1_n\\ m1_n & -\sigma\cdot p\end{pmatrix}\hat{\mu}(p-q)-\hat{\mu}(p-q)\frac{1}{E_q}\begin{pmatrix}\sigma\cdot q & m1_n\\ m1_n & -\sigma\cdot q\end{pmatrix}. \quad (2.37)$$

Here, $\hat{\mu}$ denotes the distributional Fourier transform of μ.

To state our first result, we introduce the following subalgebras of $\mathscr{g}$:

$$\mathscr{g}_\infty\equiv\{M\in\mathscr{g}\,|\,[P_+,M]\text{ is compact}\}, \quad (2.38)$$

$$\mathscr{g}_\chi\equiv\left\{M\in\mathscr{g}\,|\,\mu(x)=\begin{pmatrix}1_n\otimes\mu_+(x) & 0\\ 0 & 1_n\otimes\mu_-(x)\end{pmatrix},\mu_s(x)\in M_k(\mathbb{C})\right\}. \quad (2.39)$$

(Here, χ stands for "chiral.") Note that $\mathscr{g}_\chi$ is a W^*-algebra satisfying

$$\mathscr{g}_\chi=\mathscr{g}\cap\{\alpha_1,\ldots,\alpha_{2N-1}\}', \quad (2.40)$$

whereas $\mathscr{g}_\infty$ is a unital C^*-algebra.

Theorem 2.3. *One has*

$$\mathscr{g}_\infty\subset\mathscr{g}_\chi. \quad (2.41)$$

Proof. Assume $M\in\mathscr{g}_\infty$. Picking $e\in S^{2N-2}$, one obtains from (2.37) the distributional limit

$$\lim_{\lambda\to\infty}C_M(p+\lambda e,q+\lambda e)=\begin{pmatrix}\sigma\cdot e & 0\\ 0 & -\sigma\cdot e\end{pmatrix}\hat{\mu}(p-q)-\hat{\mu}(p-q)\begin{pmatrix}\sigma\cdot e & 0\\ 0 & -\sigma\cdot e\end{pmatrix}. \quad (2.42)$$

Invoking Lemma C2 and the compactness assumption, it follows that $\mu(x)$ commutes with $\alpha\cdot e$. In view of (2.40) this entails $M\in\mathscr{g}_\chi$. □

For $N=1$ multipliers in $\mathscr{g}_\chi$ yield off-diagonal parts that are HS, provided a Sobolev-type condition is met, cf. [2], pp. 29–30. This is in sharp contrast to the case $N>1$, as will now be shown. (For $N=2$ the following result dates back to [11].)

Theorem 2.4. *One has* $M_{\pm\mp}=0$ *if and only if*

$$\mu(x)=\begin{cases}\begin{pmatrix}1_n\otimes\lambda_+ & 0\\ 0 & 1_n\otimes\lambda_-\end{pmatrix}, & \lambda_s\in M_k(\mathbb{C}),\quad m=0\\ 1_{2n}\otimes\lambda, & \lambda\in M_k(\mathbb{C}),\quad m>0\end{cases} \quad (2.43)$$

For $N>1$, $[P_+,M]$ is HS *if and only* (2.43) *holds.*

Proof. If (2.43) holds, then P_+ commutes with M, cf. (2.37), so $[P_+,M]$ is trivially HS. Conversely, assume $[P_+,M]$ is HS. Then $M\in\mathscr{g}_\infty$, so $M\in\mathscr{g}_\chi$ by virtue of Theorem 2.3. Hence we obtain

$$C_M(p,q)=\begin{bmatrix}\Sigma(p,q)\otimes\hat{\mu}_+(p-q) & \frac{m}{E_p}\hat{\mu}_-(p-q)-\frac{m}{E_q}\hat{\mu}_+(p-q)\\ \frac{m}{E_p}\hat{\mu}_+(p-q)-\frac{m}{E_q}\hat{\mu}_-(p-q) & -\Sigma(p,q)\otimes\hat{\mu}_-(p-q)\end{bmatrix}, \quad (2.44)$$

where

$$\Sigma(p,q) \equiv \sigma\cdot\left(\frac{p}{E_p} - \frac{q}{E_q}\right). \tag{2.45}$$

The HS assumption implies that the distributions

$$T_s(p,q) \equiv \Sigma(p,q) \otimes \hat{\mu}_s(p-q), \quad s = +, - \tag{2.46}$$

have matrix elements in $L^2(\mathbb{R}^{4N-2}, dp\,dq)$. Now Σ is smooth and invertible on the set

$$S \equiv \left\{ (p,q) \in \mathbb{R}^{4N-2} \,\middle|\, p \neq 0, q \neq 0, \frac{p}{E_p} \neq \frac{q}{E_q} \right\}, \tag{2.47}$$

since

$$\Sigma^2(p,q) = \left(\frac{p}{E_p} - \frac{q}{E_q}\right)^2 1_n. \tag{2.48}$$

Hence it follows that

$$\hat{\mu}_s(p-q) \in L^2_{\mathrm{loc}}(S, dp\,dq)^{k^2}. \tag{2.49}$$

A moment's thought shows that this entails

$$\hat{\mu}_s(p) \in L^2_{\mathrm{loc}}(\mathbb{R}^{2N-1}\backslash\{0\}, dp)^{k^2}. \tag{2.50}$$

But then we must have, using (2.48),

$$\int_{p\neq q} dp\,dq \left(\frac{p}{E_p} - \frac{q}{E_q}\right)^2 \sum_{i,j=1}^{k} |\hat{\mu}_s(p-q)_{ij}|^2 < \infty. \tag{2.51}$$

Invoking Lemma C3, we infer that for $N > 1$ this implies $\hat{\mu}_s(k) = 0$, $k \neq 0$. Thus, $\hat{\mu}_s(k)$ must have support at the origin. But then the matrix elements of $\mu_s(x)$ are polynomials. Since $\mu_s(x)$ is bounded, it must be constant. The rest of the proof is obvious. □

It remains to determine compactness conditions in terms of the functions $\mu_s(x)$. The following result gives a sufficient condition.

Theorem 2.5. *Suppose $M \in \mathscr{I}_\chi$ and suppose there exist $\lambda_\pm \in M_k(\mathbb{C})$ such that the functions*

$$\alpha_s(x) \equiv \mu_s(x) - \lambda_s \tag{2.52}$$

are continuous and vanish at ∞. Then M is in $\mathscr{I}_\infty$ for $m = 0$, whereas M is in $\mathscr{I}_\infty$ for $m > 0$ if and only if $\lambda_+ = \lambda_-$.

Proof. It suffices to show $A \in \mathscr{I}_\infty$, where

$$\check{A} \equiv \begin{pmatrix} 1_n \otimes \alpha_+(\cdot) & 0 \\ 0 & 1_n \otimes \alpha_-(\cdot) \end{pmatrix}. \tag{2.53}$$

Moreover, we may take $\alpha_s(x) \in C_0^\infty(\mathbb{R}^{2N-1})^{k^2}$, since $\mathscr{I}_\infty$ is norm closed. Then $\hat{\alpha}_s(p) \in L^1(\mathbb{R}^{2N-1})^{k^2}$, so that compactness follows from Lemma C1 by noting that the matrix elements of $C_A(p,q)$ are kernels of the form (C1) with B satisfying (C2) and (C3). □

We now consider necessary conditions. It is presumably false that $\mu_\pm$ must be continuous on $\mathbb{R}^{2N-1}$ to obtain $M\in\mathscr{I}_\infty$. (For $N=1$ this follows from known results concerning Toeplitz operators, cf. [12, 13].) However, a particular kind of discontinuity to be described now does wreck compactness: We shall say that $f\in L^\infty(\mathbb{R}^l)$ has a hedge-hog discontinuity at $x_0\in\mathbb{R}^l$ if there exists a non-constant function h on S^{l-1} such that

$$\lim_{\varepsilon\to 0} f(\varepsilon(x-x_0)) = h\left(\frac{x-x_0}{|x-x_0|}\right) \quad \text{pointwise a.e..} \tag{2.54}$$

Note this amounts to a jump discontinuity when $l=1$.

Theorem 2.6. *Let $M\in\mathscr{I}_\infty\subset\mathscr{I}_\chi$. Then (the matrix elements of) $\mu_\pm$ have no hedge-hog discontinuities.*

Proof. We assume that a hedge-hog does occur for μ_+ (e.g.) and derive a contradiction. Since P_+ commutes with translations, we may assume the hedge-hog sits at the origin.

First, consider the case $m=0$. Then P_+ commutes with the scaling group $D(\varepsilon)$ and with the chiral projection $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, cf. (2.13), (2.15). Since

$$w\text{-}\lim_{\varepsilon\to 0} D(\varepsilon) = 0 \tag{2.55}$$

due to the Riemann–Lebesgue lemma, and since $\operatorname{diag}(1_n\otimes\mu_+(\cdot),0)\in\check{\mathscr{I}}_\infty$ by assumption, it follows that

$$s\text{-}\lim_{\varepsilon\to 0}\left[\check{P}_+,\begin{pmatrix}1_n\otimes\mu_+^\varepsilon(\cdot) & 0\\ 0 & 0\end{pmatrix}\right]=0,\quad \mu_+^\varepsilon(x)\equiv\mu_+(\varepsilon x). \tag{2.56}$$

Moreover, by assumption we have pointwise a.e.

$$\lim_{\varepsilon\to 0}\mu_+^\varepsilon(x) = h\left(\frac{x}{|x|}\right), \tag{2.57}$$

where h is a non-constant matrix-valued function on S^{2N-2}. By dominated convergence, (2.57) also holds in the sense of strong convergence of bounded multipliers on $L^2(\mathbb{R}^{2N-1},dx)^k$. But then (2.56) entails

$$\left[\check{P}_+,\begin{pmatrix}1_n\otimes h(\cdot) & 0\\ 0 & 0\end{pmatrix}\right]=0. \tag{2.58}$$

Invoking now Theorem 2.4, we conclude h is constant a.e., which is a contradiction.

Next, we take $m>0$. From Theorem 2.5 we have

$$\begin{pmatrix}1_n\otimes\phi(\cdot) & 0\\ 0 & 0\end{pmatrix}\in\check{\mathscr{I}}_\infty,\quad \phi\in C_0^\infty,\quad \phi(0)=1_k, \tag{2.59}$$

so we may assume $\operatorname{supp}\mu_+$ is compact and $\mu_-=0$. Then we have $\mu_+\in L^r$, $\forall r\in[1,\infty]$. Now it is easy to check

$$P_{(m)+}(p)-P_{(o)+}(p)\in L^r,\quad \forall r>2N-1, \tag{2.60}$$

where the dependence on the mass is explicitly indicated. Hence, we may use Theorem XI.20 in [7] to infer $[P_{(m)+}-P_{(o)+},M]$ is compact. Since $[P_{(m)+},M]$ is compact by assumption, it follows that $[P_{(o)+},M]$ is compact. Thus we obtain the desired contradiction. □

Next, we study the behavior at ∞. We say that $f\in L^{\infty}(\mathbb{R}^l)$ has a hedge-hog discontinuity at ∞ if there exists a non-constant function h on S^{l-1} such that

$$\lim_{\varepsilon\to\infty} f(\varepsilon x)=h\left(\frac{x}{|x|}\right) \quad \text{pointwise a.e..} \tag{2.61}$$

Now a distinction arises between the cases $m=0$ and $m>0$.

Theorem 2.7. *Let $m=0$ and assume $M\in\mathcal{g}_{\infty}\subset\mathcal{g}_{\chi}$. Then $\mu_{\pm}(x)$ have no hedge-hog discontinuity at ∞. Now let $m>0$ and assume $\mu_s(x)$ are continuous on $\mathbb{R}^{2N-1}$; moreover, suppose a continuous function h on S^{2N-2} exists such that*

$$\mu_s(x)-h\left(\frac{x}{|x|}\right)=o(1), \quad |x|\to\infty, \quad s=+,-. \tag{2.62}$$

Then $M\in\mathcal{g}_{\infty}$.

Proof. The $m=0$ assertion follows by arguing as in the proof of Theorem 2.6, taking $\varepsilon\to\infty$ instead of $\varepsilon\to 0$. To prove the $m>0$ claim, it suffices to show $M\in\mathcal{g}_{\infty}$ for $\mu_s(x)$ of the form

$$\mu_s(x)=\phi(|x|)h\left(\frac{x}{|x|}\right), \quad h\in C(S^{2N-2})^{k^2}, \quad \phi\in C([0,\infty)), \quad \phi(r)=\begin{cases}0 & r<1\\ 1 & r>2\end{cases}. \tag{2.63}$$

The product of two such functions is again of this form, and this is also true for the sum, provided $\phi_1=\phi_2$. Thus, if we can show $M\in\mathcal{g}_{\infty}$ when h is equal to the product of an arbitrary $\lambda\in M_k(\mathbb{C})$ and one of the functions $x_1/|x|,\ldots,x_{2N-1}/|x|$, then it follows that $M\in\mathcal{g}_{\infty}$ when h is a matrix whose elements are polynomials in these functions. By the Stone–Weierstrass theorem such polynomials are uniformly dense in the continuous functions on S^{2N-2}. Therefore, we may infer $M\in\mathcal{g}_{\infty}$ when (2.63) holds.

The upshot is, that we need only consider the multipliers $\check{M}_i$ for which

$$\mu_{s,i}(x)=\frac{x_i}{(1+|x|^2)^{1/2}}\lambda, \quad \lambda\in M_k(\mathbb{C}), \quad s=+,-, \quad i=1,\ldots,2N-1. \tag{2.64}$$

Here, we have replaced $\phi(r)$ by the function $r/(1+r^2)^{1/2}$, the difference being a continuous function vanishing at ∞ and at 0. The point of this replacement is, that the Fourier transform of the functions

$$a_i(x)\equiv x_i/(1+|x|^2)^{1/2} \tag{2.65}$$

can be found explicitly. Indeed, we have

$$\int_{\mathbb{R}} dx\frac{\exp(ipx)}{(1+x^2)^{1/2}}\sim K_o(|p|)=\begin{cases}O(\ln|p|), & p\to 0\\ O(\exp(-|p|)), & |p|\to\infty\end{cases}, \tag{2.66}$$

and for $N>1$ we can use

$$\int_{\mathbb{R}^{2N-1}} dx \frac{\exp(ipx - \varepsilon|x|)}{(1+|x|^2)^{1/2}} \sim \frac{1}{|p|} \partial_{|p|}^{2N-3} \int_0^\infty dr \frac{\cos|p|r \exp(-\varepsilon r)}{(1+r^2)^{1/2}} \tag{2.67}$$

to infer

$$\mathscr{F}(1/(1+|x|^2)^{1/2}) \sim \frac{1}{|p|} \partial_{|p|}^{2N-3} K_0(|p|) = \begin{cases} O(|p|^{-2N+2}), & p \to 0 \\ O(\exp(-|p|)), & |p| \to \infty \end{cases} \tag{2.68}$$

(cf. e.g. [14]). The Fourier transform of $a_i(x)$ is then obtained by taking the i^{th} partial derivative of (2.68) in the weak sense. This yields a distribution $\hat{a}_i(p)$ that equals a smooth function on $\mathbb{R}^{2N-1}\backslash\{0\}$. This function has a non-integrable singularity at 0, but multiplication by $p_j, j \in \{1, \ldots, 2N-1\}$, suffices to render it integrable at 0. The result is then an L^1-function $\hat{a}_{ij}(p)$ which is easily seen to equal the distribution $p_j \hat{a}_i(p)$.

The crux is now that (2.64) entails

$$C_{M_i}(p,q) = \sum_{j=1}^{2N-1} \hat{a}_{ij}(p-q) B_j(p,q) \otimes \lambda, \tag{2.69}$$

where the matrix elements of $B_1, \ldots, B_{2N-1}$ satisfy (C2) and (C3). Indeed, let us set

$$B(p,q) \equiv \begin{bmatrix} \sigma \cdot \left(\frac{p}{E_p} - \frac{q}{E_q} \right) & m\left(\frac{1}{E_p} - \frac{1}{E_q} \right) \\ m\left(\frac{1}{E_p} - \frac{1}{E_q} \right) & -\sigma \cdot \left(\frac{p}{E_p} - \frac{q}{E_q} \right) \end{bmatrix}. \tag{2.70}$$

Using the Taylor expansion

$$B(p,q) = B(p,p) + \int_0^1 ds \sum_{j=1}^{2N-1} (q_j - p_j)(\partial_{q_j} B)(p, p + s(q-p))$$

and $B(p,p) = 0$, we then obtain

$$B_j(p,q) = \int_0^1 ds \begin{bmatrix} -\frac{\sigma_j}{E_q} + \frac{q_j}{E_q^3}(\sigma \cdot q) & \frac{m q_j}{E_q^3} \\ \frac{m q_j}{E_q^3} & \frac{\sigma_j}{E_q} - \frac{q_j}{E_q^3}(\sigma . q) \end{bmatrix}_{q = p + s(q-p)} \tag{2.71}$$

from which the assertion readily follows. Because $\hat{a}_{jk} \in L^1$, we can invoke Lemma C1 to complete the proof. □

2.4. Unitary Multipliers. In this subsection we study the groups G, G_χ and G_∞ obtained from $\mathscr{g}, \mathscr{g}_\chi$ and $\mathscr{g}_\infty$ by restricting to operators of the form

$$(\check{U} f)(x) = u(x) f(x), \quad u(x) \in U(2nk), \quad f \in \check{\mathscr{H}}. \tag{2.72}$$

(For most of what follows one can just as well consider multipliers with bounded inverses.) We start with some simple observations concerning Fredholm properties of the diagonal parts of U. First, using unitarity we conclude

$$U \in G_\infty \Leftrightarrow U_{\pm\pm} \text{ are essentially unitary.} \tag{2.73}$$

(Of course, this is meant in the sense of operators on $\mathscr{H}_\pm$, respectively.) In particular,

$U_{\delta\delta}$ are Fredholm when $U \in G_\infty$. However, it is clear from Theorem 2.2 that there exist smooth $u(x)$ with limit 1 for $|x| \to \infty$ such that $U_{\delta\delta}$ are not Fredholm.

On the other hand, $U_{\delta\delta}$ can be Fredholm without U belonging to G_∞. Indeed, one need only pick $U \notin G_\chi$ (and hence $U \notin G_\infty$, cf. Theorem 2.3) whose eigenvalues stay in a sector

$$S_\phi \equiv \{z \in \mathbb{C} \mid |\mathrm{Arg} z| < \phi\} \tag{2.74}$$

with $\phi < \pi/2$. For any unit vector f in $\mathscr{H}_\delta$ one then gets

$$\| U_{\delta\delta} f \| \geqq |(f, U_{\delta\delta} f)| \geqq \mathrm{Re}(f, Uf) \geqq \cos\phi, \tag{2.75}$$

and hence

$$\| U_{\delta\delta}{}^{-1} \| \leqq (\cos\phi)^{-1} \quad \phi \in \left[0, \frac{\pi}{2}\right). \tag{2.76}$$

We continue by introducing a subgroup G_e of G_χ whose elements have an obvious topological interpretation. By definition, $U \in G_e$ if and only if the functions $u_\pm(x) \in U(k)$ are continuous on $\mathbb{R}^{2N-1}$ and satisfy

$$u_s(x) - 1_k = o(1), \quad |x| \to \infty, \quad s = +, -. \tag{2.77}$$

Thus, $u_\pm(x)$ may be viewed as continuous maps from S^{2N-1} to $U(k)$, reducing to 1_k at the north pole. (Here and from now on, we view $\mathbb{R}^{2N-1}$ as arising from S^{2N-1} by stereographic projection.) In view of Theorem 2.5 we have $G_e \subset G_\infty$, so $U_{\delta\delta}$ are Fredholm when $U \in G_e$.

Now it is clear that the standard kink

$$\check{K} \equiv \begin{bmatrix} 1_n \otimes \dfrac{\sigma \cdot x + (-)^N i}{\sigma \cdot x - (-)^N i} & 0 \\ 0 & 1_n \otimes 1_n \end{bmatrix} \tag{2.78}$$

of Subsect. 2.2 belongs to $\check{G}_e$ (taking $k = n$). Also, it follows from Theorem 2.1 that

$$\mathrm{index}\, K_{--} \equiv \dim \mathrm{Ker}\, K_{--} - \dim \mathrm{Ker}\, K^*_{--} = 1. \tag{2.79}$$

Since the Fredholm index is norm continuous, the kink map

$$u_K : S^{2N-1} \to U(n), \quad x \in \mathbb{R}^{2N-1} \subset S^{2N-1} \mapsto \frac{\sigma \cdot x + (-)^N i}{\sigma \cdot x - (-)^N i} \tag{2.80}$$

cannot be null homotopic.

Next, recall that by virtue of Bott's periodicity theorem one has

$$\pi_{2N-1}(U(k)) = \mathbb{Z}, \quad \pi_{2N-2}(U(k)) = 0, \quad k \geqq N \tag{2.81}$$

(cf. e.g. [15] and references given there). By convention we shall choose the "winding number" $w \in \mathbb{Z}$ of u_K positive. We claim this implies

$$w(u_K) = 1. \tag{2.82}$$

Indeed, assuming $w > 1$, there would exist a continuous map u_1 with u_1^w homotopic to u_K. This would imply index $K_{--} \in w\mathbb{Z}$, which contradicts (2.79). Thus, the kink map is a homotopy generator. (This can also be seen directly; in fact, u_K is in essence the map a_N defined in [15], p. 228.) We are now in a position to state the following theorem, which is one of the main results of this paper.

Theorem 2.8. *One has* $G_e \subset G_\infty$ *and for* $k \geqq N$,

$$\operatorname{index} U_{--} = w(u_+) - w(u_-) \quad \forall U \in G_e. \tag{2.83}$$

Proof. We have already proved the inclusion, cf. Theorem 2.5. Picking $U \in G_e$ we have

$$\check{U} = \begin{pmatrix} 1_n \otimes u_+(\cdot) & 0 \\ 0 & 1_n \otimes u_-(\cdot) \end{pmatrix} = \begin{pmatrix} 1_n \otimes u_+(\cdot) & 0 \\ 0 & 1_n \otimes 1_k \end{pmatrix} \begin{pmatrix} 1_n \otimes 1_k & 0 \\ 0 & 1_n \otimes u_-(\cdot) \end{pmatrix} \equiv \check{U}_+ \check{U}_-, \tag{2.84}$$

and hence

$$\operatorname{index} U_{--} = \operatorname{index} U_{+,--} + \operatorname{index} U_{-,--}. \tag{2.85}$$

Furthermore, a continuous map

$$u: S^{2N-1} \to U(k), \quad x \mapsto u(x), \quad k \geqq N \tag{2.86}$$

has the same winding number as the maps

$$u_l: S^{2N-1} \to U(k+l), \quad x \mapsto u(x) \otimes 1_l, \quad \forall l > 0.$$

It is also obvious that the Fredholm indices of the corresponding operators are equal. Using all this, the index formula (2.83) readily follows from its validity for the standard kink K (cf. (2.79), (2.82)) and its parity transform $K_{-,1}$ (cf. (2.18)). □

In the remainder of this subsection we take $m > 0$. We shall consider continuous multipliers in G_χ for which there exists $u_\infty \in C(S^{2N-2}, U(k))$ with

$$u_s(x) - u_\infty\left(\frac{x}{|x|}\right) = o(1), \quad |x| \to \infty, \quad s = +, -. \tag{2.87}$$

On account of Theorem 2.7 such multipliers form a subgroup of G_∞, denoted G_h. Clearly, the map $u_+ u_-^{-1}$ is continuous at ∞, and hence has a well-defined winding number $w \in \mathbb{Z}$ when $k \geqq N$, cf. (2.81). This prepares us for our next result.

Theorem 2.9. *Let* $U \in G_h$ *and* $k \geqq N$. *Then one has*

$$\operatorname{index} U_{--} = w(u_+ u_-^{-1}). \tag{2.88}$$

Proof. For $k \geqq N$ the map u_∞ is null homotopic in view of (2.81). Thus, a continuous map

$$u(t, \Omega): [0,1] \times S^{2N-2} \to U(k), \quad u(1, \Omega) = u_\infty(\Omega), \quad u(0, \Omega) = 1_k \tag{2.89}$$

exists. Fixing $T \in [0,1]$ we define a map

$$u_T: \mathbb{R}^{2N-1} \to U(k), \quad x \mapsto \begin{cases} u\left(|x|, \dfrac{x}{|x|}\right), & |x| < T \\ u\left(T, \dfrac{x}{|x|}\right), & |x| \geqq T \end{cases} \tag{2.90}$$

and $U_T \in G_h$ by

$$\check{U}_T \equiv 1_{2n} \otimes u_T(\cdot). \tag{2.91}$$

Then it follows that $U U_1^{-1} \in G_e$ and hence, using Theorem 2.8,

$$\begin{aligned}\text{index}\, U_{--} - \text{index}\, U_{1--} &= w(u_+ u_1^{-1}) - w(u_- u_1^{-1}) \\ &= w(u_+ u_1^{-1}(u_- u_1^{-1})^{-1}) = w(u_+ u_-^{-1}).\end{aligned} \tag{2.92}$$

Thus it remains to show index $U_{1--} = 0$. But this follows from the easily verified fact that U_T is norm continuous in T on $[0, 1]$. □

For $k = n$ there exists a family $H_\varepsilon \in G_h$, $\varepsilon \in (0, \infty)$, for which index $H_{\varepsilon--}$ can be determined without invoking Bott periodicity. This "standard hedge-hog family" is defined by

$$\check{H}_\varepsilon \equiv \frac{1}{(|x|^2 + \varepsilon^2)^{1/2}} \begin{pmatrix} 1_n \otimes (\sigma \cdot x + (-)^N i\varepsilon) & 0 \\ 0 & 1_n \otimes (\sigma \cdot x - (-)^N i\varepsilon) \end{pmatrix}. \tag{2.93}$$

The point is the simple relation

$$H_\varepsilon^2 = K_{+,\varepsilon} K_{-,\varepsilon} \tag{2.94}$$

with the standard kinks: It entails

$$\text{index}\, H_{\varepsilon--} = 1, \quad \forall \varepsilon \in (0, \infty), \tag{2.95}$$

since $K_{s,\varepsilon}$ has index 1. We also observe that one only needs the last part of the proof of Theorem 2.7 to prove that $H_{\varepsilon \pm \mp}$ are compact. Indeed, the proof of this theorem hinges on reducing the general case to a special case which arises precisely for the standard hedge-hogs, cf. (2.64).

3. Approximate Quantum Fields

3.1. Preliminaries. So far, we have not had occasion to use the positive and negative energy Dirac spinors in terms of which the Dirac and Majorana fields occur in the physics literature. However, in Subsects. 3.2 and 3.3 we aim to elucidate the intimate relation of these fields to fermion Fock space quadratic forms associated with the standard kinks. Therefore, we shall in this subsection elaborate on the classical (single particle) context as presented in Subsect. 2.1, in preparation for the Dirac and Majorana quantizations to be described below. We again take $k = 1$ at first, so as to ease the notation.

We shall work with Dirac spinors

$$w_\delta^j(\delta p) \in \mathbb{C}^{2n}, \quad \delta = +, -, \quad j = 1, \ldots, n, \quad p \in \mathbb{R}^{2N-1}, \tag{3.1}$$

yielding orthonormal bases for the positive and negative energy subspaces of the matrix multiplier $H(p)$, so that

$$H(p) w_\delta^j(\delta p) = \delta E_p w_\delta^j(\delta p) \tag{3.2}$$

cf. (2.4). The positive energy spinors are defined by

$$w_+^j(p) \equiv \left(\frac{2E_p}{E_p+m}\right)^{1/2} P_+(p)b_j, \tag{3.3}$$

where b_j are the unit vectors with components

$$(b_j)_l \equiv 2^{-1/2}(\delta_{j,l} + \delta_{j+n,l}), \quad j = 1,\dots,n, \quad l = 1,\dots,2n. \tag{3.4}$$

Using (2.6) and (2.4) it is readily seen that these spinors are indeed orthonormal. The negative energy spinors are now given by

$$w_-^j(p) \equiv U_c \overline{w_+^j(p)}, \quad j = 1,\dots,n. \tag{3.5}$$

(As before, the bar denotes complex conjugation and not the Pauli adjoint.) The relations (A43) defining U_c imply that the spinors $w_-^j(-p)$ yield an orthonormal base for the negative eigenvalue subspace of $H(p)$, as promised.

Using these spinors we can now transform to a spectral representation for $\check{H}$ on

$$L^2(\mathbb{R}^{2N-1}, dp)^n \oplus L^2(\mathbb{R}^{2N-1}, dp)^n \tag{3.6}$$

in the sense that the transform of $\check{H}$ acts as multiplication by $E_p 1_n \oplus -E_p 1_n$ on this space. Of course, (3.6) is just the space $\mathscr{H}$ of Sect. 2, looked at from another perspective. We shall use the following device in an attempt to simultaneously prevent confusion and ease the notation: The space $\mathscr{H}$ of Sect. 2 and operators A acting on it will be denoted $\check{\mathscr{H}}$ and $\check{A}$ from now on, whereas the notation $\mathscr{H}$ and A will be reserved for the spectral representation space (3.6) and operators acting on it.

Explicitly, the representation is set up by the unitary operator

$$W: \mathscr{H} \to \check{\mathscr{H}}, \quad g \mapsto \mathscr{F}^{-1}\mathscr{D}^{-1}g, \tag{3.7}$$

where $\mathscr{F}$ is Fourier transformation, cf. (2.2), and $\mathscr{D}: \hat{\mathscr{H}} \to \mathscr{H}$ is the diagonalizing transformation

$$(\mathscr{D}f)_\delta^j(p) \equiv \overline{w_\delta^j(p)} \cdot f(\delta p), \quad \delta = +,-, \quad j = 1,\dots,n, \tag{3.8}$$

whose inverse reads

$$(\mathscr{D}^{-1}g)(p) = \sum_{\delta,j} g_\delta^j(\delta p) w_\delta^j(\delta p). \tag{3.9}$$

Then one gets

$$(Hf)_\delta(p) = \delta E_p f_\delta(p) \tag{3.10}$$

as announced. (We suppress the superscript j whenever it is not acted on.)

In the next two subsections we shall employ the customary Euclidean group representation of the one-particle Dirac theory. Its action on $\check{\mathscr{H}}$ reads

$$(\check{U}(a,R)f)(x) \equiv S(R)f(R^{-1}(x-a)), \quad a \in \mathbb{R}^{2N-1}, \quad R \in SO(2N-1). \tag{3.11}$$

(The spinor representation $S(\cdot)$ of $SO(2N-1)$ is defined in Appendix A.) Using (A29) one gets

$$[U(a,R), H] = 0, \quad [U(a,R), P_\delta] = 0. \tag{3.12}$$

We continue by introducing charge conjugation, which plays a crucial role in Subsect. 3.3. It reads

$$(\check{C}f)(x) \equiv U_c \bar{f}(x) \tag{3.13}$$

and satisfies

$$[C, U(a,R)] = 0 \tag{3.14}$$

due to (A50). Moreover, (A43) entails

$$CH = -HC, \quad CP_\delta = P_{-\delta}C, \tag{3.15}$$

and from (A46)–(A49) we have

$$C^2 = \pm 1, \quad N \equiv \begin{cases} 1,2 \\ 3,0 \end{cases} \bmod 4. \tag{3.16}$$

The properties of U_c can also be used to calculate the transforms of $\check{P}$ (cf. (2.10)) and $\check{C}$ to $\mathscr{H}$: These are given by

$$(Pf)_\delta(p) = \delta f_\delta(-p) \tag{3.17}$$

$$(Cf)_\delta(p) = \begin{cases} \bar{f}_{-\delta}(p) \\ -\delta \bar{f}_{-\delta}(p) \end{cases} \quad N \equiv \begin{cases} 1,2 \\ 3,0 \end{cases} \bmod 4. \tag{3.18}$$

From now on we again assume that an internal symmetry space $\mathbb{C}^k$ is tensored on to $\mathscr{H}$. As before, we keep the same notation and note that then all of the above relations still hold.

3.2. Approximate Dirac Fields. The free Dirac field is a $\mathbb{C}^{2n} \otimes \mathbb{C}^k$-valued quadratic form on the fermion Fock space $\mathscr{F}_a(\mathscr{H}) \cong \mathscr{F}_a(\mathscr{H}_+) \otimes \mathscr{F}_a(\mathscr{H}_-)$, defined by

$$\psi(t,x) \equiv (2\pi)^{-(2N-1)/2} \sum_{\substack{j=1,\ldots,n \\ l=1,\ldots,k}} \int dp[a_{j,l}(p) w_+^j(p) \otimes e_l \exp(-iE_p t + ip\cdot x)$$

$$+ b_{j,l}^*(p) w_-^j(p) \otimes e_l \exp(iE_p t - ip\cdot x)]. \tag{3.19}$$

Here, $\{e_1, \ldots, e_k\}$ is the canonical basis of $\mathbb{C}^k$. Since the functions involved are bounded, we may and shall choose as form domain the dense subspace $\mathscr{D}_{at,0}^\infty$ of algebraic tensors whose constituent functions are in C_0^∞. One readily verifies

$$\int dx \bar{g}(x) \cdot \psi(t,x) = \Phi(\exp(itH)W^{-1}g), \quad \forall g \in \check{\mathscr{H}}, \tag{3.20}$$

where

$$\Phi(f) \equiv a(P_+ f) + b^*(\overline{P_- f}), \quad f \in \mathscr{H} \tag{3.21}$$

is the "abstract" Dirac field. This smeared field satisfies the CAR

$$\{\Phi(f), \Phi(g)\} = 0, \quad \{\Phi(f), \Phi(g)^*\} = (f,g), \tag{3.22}$$

and hence the transformation $\Phi(f) \to \Phi(Uf)$ yields an automorphism of the CAR (Bogoliubov transformation) provided U is unitary.

It is well known that such a transformation can be unitarily implemented if and only if the off-diagonal parts $U_{\pm\mp}$ are Hilbert–Schmidt. Moreover, the structure of the unitary implementer is known [16, 17]. It involves a multiplicative factor that is in essence an infinite determinant. Omitting this factor yields an operator $\tilde{\Gamma}_r(U)$ (r for renormalized) which is expressed in terms of an operator Z. When the HS condition is violated, this expression no longer defines an operator. However, it still makes rigorous sense as a quadratic form on the subspace $\mathscr{D}_{at}$ of algebraic tensors, provided the diagonal parts $U_{\pm\pm}$ are Fredholm.

We have already seen that for $N>1$ the unitaries of interest to us, viz., the multipliers of Subsect. 2.4, are trivial when one insists on the HS property, cf. Theorem 2.4. However, multipliers in G_∞ do have the Fredholm property, and in particular the standard kinks $K_{s,\varepsilon}$ of Subsect. 2.2 have Fredholm diagonal parts. Therefore, they give rise to well-defined quadratic forms on $\mathscr{D}_{at}$.

We shall take

$$\mathbb{C}^k = \mathbb{C}^n \tag{3.23}$$

from now on. If $k>n$ one can obtain analogous results, but $k=n$ is the minimum value for which we can construct approximate Dirac fields, since we have no explicit information on winding-number-one unitaries in G_∞ for $k<n$. The approximate Dirac fields are expressed in terms of the form implementers of the Bogoliubov transformations generated by the standard kinks. Explicitly, we may and shall take

$$\tilde{\Gamma}_r(K_{s,\varepsilon}) \equiv a^*(\rho_{s,\varepsilon,+})E_c(Z_{s,\varepsilon}) + E_c(Z_{s,\varepsilon})b(\bar{\rho}_{s,\varepsilon,-}). \tag{3.24}$$

Here, the operators $Z_{s,\varepsilon}$ are the kink conjugates defined in Appendix D, and

$$\rho_{s,\varepsilon,\delta} \equiv (2\pi)^{-(2N-1)/2}\mathscr{D}\kappa_{s,\varepsilon,\delta} \tag{3.25}$$

cf. (2.20), (2.22); the norm of the kernel functions is chosen with an eye on what follows. Furthermore,

$$E_c(Z) \equiv \exp(Z_{+-}a^*b^*)\Gamma(Z_{++}\oplus Z^T_{--})\exp(-Z_{-+}ba), \tag{3.26}$$

where $\Gamma(\cdot)$ denotes the Fock space product operation.

Next, we introduce the forms

$$\psi^*_{s,\varepsilon}(a,R) \equiv \tilde{\Gamma}(U(a,R))\tilde{\Gamma}_r(K_{s,\varepsilon})\Gamma(-1)\tilde{\Gamma}(U(a,R))^*, \quad a\in\mathbb{R}^{2N-1},\ R\in SO(2N-1) \tag{3.27}$$

and their adjoints $\psi_{s,\varepsilon}(a,R)$ with form domain $\mathscr{D}_{at}$. (This is well defined: $\mathscr{D}_{at}$ is left invariant by $\tilde{\Gamma}(U)$ when $U_{\pm\mp}=0$ and this is the case here, cf. (3.12).) Moreover, we set

$$\psi_+(a,R) \equiv \begin{pmatrix} \bar{u}_R \\ 0 \end{pmatrix}\cdot\psi(0,Ra), \tag{3.28}$$

$$\psi_-(a,R) \equiv \begin{pmatrix} 0 \\ \bar{u}_R \end{pmatrix}\cdot\psi(0,Ra), \tag{3.29}$$

where u_R is defined in Appendix A, cf. (A35). Using (3.20) we then have, e.g.,

$$\psi_+(f,R) \equiv \int da\bar{f}(Ra)\psi_+(a,R) = \Phi\left(W^{-1}\begin{pmatrix} fu_R \\ 0 \end{pmatrix}\right), \quad f\in L^2(\mathbb{R}^{2N-1}). \tag{3.30}$$

By virtue of Lemma A3 this implies that the smeared fields

$$\{\psi_s^{(*)}(f,R) \mid f \in L^2(\mathbb{R}^{2N-1}), \quad R \in SO(2N-1)\} \tag{3.31}$$

act irreducibly in $\mathscr{F}_a(\mathscr{H})$. We are now in a position to present a principal result of this paper, showing that the forms $\psi_{s,\varepsilon}^{(*)}(a,R)$ may be viewed as approximate Dirac fields.

Theorem 3.1. *One has*

$$\lim_{\varepsilon \to 0} \psi_{s,\varepsilon}^{(*)}(a,R) = \psi_s^{(*)}(a,R) \tag{3.32}$$

in the sense of quadratic forms on $\mathscr{D}_{at,0}^{\infty}$.

Proof. From (3.28)–(3.30) and (3.20) it follows that

$$\psi_s^{(*)}(a,R) = \tilde{\Gamma}(U(a,R))\psi_s^{(*)}(0,1)\tilde{\Gamma}(U(a,R))^*, \tag{3.33}$$

so that we need only prove this for $a = 0$ and $R = 1$, cf. (3.27). Also, we need only detail the case $s = +$, since the case $s = -$ then follows by using parity. Evaluation of

$$\langle F, \tilde{\Gamma}_r(K_{+,\varepsilon})G\rangle, \quad F, G \in \mathscr{D}_{at,0}^{\infty} \tag{3.34}$$

yields a finite sum of products of terms that are inner products in $\mathscr{H}$, so that we need only determine the $\varepsilon \to 0$ behavior of these terms. Four types occur, viz.,

$$(f,g), \quad (f, Z_{+,\varepsilon,\delta\delta}g), \quad (f, Z_{+,\varepsilon,\delta-\delta}g), \quad (f, \rho_{+,\varepsilon,\delta}), \tag{3.35}$$

where $f, g \in C_0^{\infty}(\mathbb{R}^{2N-1})^{n^2}$. Each product contains one and only one term of type 4, and using dominated convergence, (3.25) and (2.20), (2.22) one infers

$$\lim_{\varepsilon \to 0} (f, \rho_{+,\varepsilon,\delta}) = (2\pi)^{-(2N-1)/2}\delta \sum_{i_1,i_2=1}^{n} \int dp \bar{f}^{i_1,i_2}(p)\overline{w_\delta^{i_1}}(p) \otimes e_{i_2} \cdot \begin{pmatrix} u \\ 0 \end{pmatrix}. \tag{3.36}$$

Also, invoking Lemma D1, one sees that type 2 and 3 inner products converge to $-(f,g)$ and 0, respectively. Using these facts it now follows that

$$\lim_{\varepsilon \to 0} \psi_{+,\varepsilon}^*(0,1) = \psi^*(0,0) \cdot \begin{pmatrix} u \\ 0 \end{pmatrix} = \psi_+^*(0,1). \tag{3.37}$$

Indeed, one need only check that the factor $\Gamma(-1)$ corrects signs where needed. Taking the form adjoint of (3.37) completes the proof of the theorem. □

3.3. Approximate Majorana Fields. The Majorana field is a $\mathbb{C}^{2n} \otimes \mathbb{C}^k$-valued quadratic form on $\mathscr{F}_a(\mathscr{H}_+)$, given by (3.19) with $a \to c$ and $b^* \to c^*$. Its form domain is defined just as in the charged case, and will be again denoted $\mathscr{D}_{at,0}^{\infty}$. Now one gets

$$\int dx \bar{g}(x) \cdot \psi(t,x) = B(\exp(itH)W^{-1}g), \quad \forall g \in \check{\mathscr{H}}, \tag{3.38}$$

where

$$B(f) \equiv c(P_+ f) \pm c^*(CP_- f), \quad N \equiv \begin{cases} 1,2 \\ 3,0 \end{cases} \bmod 4, \tag{3.39}$$

cf. (3.18). The "abstract" Majorana field B clearly satisfies

$$\{B(f), B(g)^*\} = (f, g), \quad \forall f, g \in \mathcal{H}. \tag{3.40}$$

Moreover, using (3.16) one obtains

$$B(f)^* = B(Cf), \quad N \equiv 1, 2 \bmod 4. \tag{3.41}$$

These two relations give rise to a C^*-algebra, the so-called self-dual CAR algebra [18], which may also be viewed as a complex Clifford algebra. The second relation is the smeared version of the form equality

$$\psi(t, x)^* = \check{U}_c \psi(t, x). \tag{3.42}$$

For $N \equiv 3, 0 \bmod 4$ no such local relation for ψ^* in terms of ψ exists. Moreover, (3.41) is replaced by

$$B(f)^* = B(C(P_+ - P_-)f), \quad N \equiv 3, 0 \bmod 4, \tag{3.43}$$

cf. (3.18). Correspondingly, one again gets a self-dual CAR-algebra, the conjugation with square 1 now being $C(P_+ - P_-)$.

The transformation $B(f) \to B(Uf)$ yields an automorphism of these algebras, provided U is unitary and satisfies

$$CU = UC, \quad N \equiv 1, 2 \quad \bmod 4, \tag{3.44}$$

$$C(P_+ - P_-)U = UC(P_+ - P_-), \quad N \equiv 3, 0 \bmod 4. \tag{3.45}$$

Now it is readily verified that a unitary multiplier of the form

$$\check{U} = \begin{pmatrix} 1_n \otimes u_+(\cdot) & 0 \\ 0 & 1_n \otimes u_-(\cdot) \end{pmatrix}, \quad u_\pm(x) \in U(k) \tag{3.46}$$

commutes with $\check{C}$ provided

$$u_\pm(x) \in O(k), \quad N \equiv 1 \bmod 4, \tag{3.47}$$

$$u_-(x) = \overline{u_+}(x), \quad N \equiv 2 \bmod 4, \tag{3.48}$$

cf. (A46), (A47). However, for $N \equiv 3, 0 \bmod 4$ no non-trivial multipliers commuting with $\check{C}(\check{P}_+ - \check{P}_-)$ appear to exist, since the action of $\check{P}_+ - \check{P}_-$ is non-local. Therefore, we shall henceforth restrict ourselves to a consideration of the cases $N \equiv 1, 2 \bmod 4$.

First, let $N \equiv 1 \bmod 4$. Then we take

$$\mathbb{C}^k \cong \mathbb{C}^n \otimes \mathbb{C}^2. \tag{3.49}$$

(Generalizing what follows to the case $\mathbb{C}^2 \to \mathbb{C}^l$ is a matter of bookkeeping when $l > 2$, cf. [2], where $N = 1$ and $l \geqq 2$. However, we see no way to get similar results for $l = 1$.) We now define neutral kinks, using notation that will be clear from context:

$$\check{K}^n_{+,\varepsilon} \equiv \check{M} 1_n \otimes \operatorname{diag}\left(\frac{\sigma \cdot x - i\varepsilon}{\sigma \cdot x + i\varepsilon}, \frac{\bar{\sigma} \cdot x + i\varepsilon}{\bar{\sigma} \cdot x - i\varepsilon}, 1_n, 1_n \right) \check{M}^*, \tag{3.50}$$

$$\check{K}^n_{-,\varepsilon} \equiv \check{M} 1_n \otimes \operatorname{diag}\left(1_n, 1_n, \frac{\sigma \cdot x + i\varepsilon}{\sigma \cdot x - i\varepsilon}, \frac{\bar{\sigma} \cdot x - i\varepsilon}{\bar{\sigma} \cdot x + i\varepsilon} \right) \check{M}^*, \tag{3.51}$$

$$\check{M} \equiv 1_{2n} \otimes 1_n \otimes 2^{-1/2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \tag{3.52}$$

Then it is easy to check that the matrices at the right-hand side of (3.50), (3.51) are real for any $x \in \mathbb{R}^{2N-1}$. Hence, $K^n_{s,\varepsilon}$ commutes with C. Moreover, since M commutes with P_δ, the relevant properties of the neutral kinks can be read off from the above results on the charged kinks. Specifically,

$$\kappa_{s,\varepsilon}(p) \equiv \kappa_{s,\varepsilon,+}(p) \otimes 2^{-1/2} \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{3.53}$$

spans $\operatorname{Ker} \hat{K}^{n*}_{s,\varepsilon++}$, and

$$\kappa'_{s,\varepsilon} \equiv \hat{C} \hat{K}^{n*}_{s,\varepsilon} \kappa_{s,\varepsilon} = \hat{C}\left(\kappa_{s,\varepsilon,-} \otimes 2^{-1/2} \begin{pmatrix} 1 \\ i \end{pmatrix} \right) \tag{3.54}$$

spans $\operatorname{Ker} \hat{K}^n_{s,\varepsilon++}$, cf. Subsect. 2.2; moreover, the operators $K^{n}_{s,\varepsilon}{}^{(*)}_{\pm\pm}$ are essentially unitary, cf. Subsect. 2.4.

These properties suffice to conclude that the neutral Bogoliubov transformation $B(f) \to B(K^n_{s,\varepsilon} f)$ can be implemented in form sense by

$$\hat{\Gamma}_r(K^n_{s,\varepsilon}) \equiv c^*(\rho_{s,\varepsilon}) E_n(Z^n_{s,\varepsilon}) + E_n(Z^n_{s,\varepsilon}) c(\rho'_{s,\varepsilon}). \tag{3.55}$$

Here, one has

$$\rho^{(\prime)}_{s,\varepsilon} \equiv (2\pi)^{-(2N-1)/2} \mathscr{D} \kappa^{(\prime)}_{s,\varepsilon}, \tag{3.56}$$

$$E_n(Z) \equiv \exp\left(\tfrac{1}{2} Z_{+-} c^* c^*\right) \Gamma(Z_{++}) \exp\left(-\tfrac{1}{2} Z_{-+} cc\right) \tag{3.57}$$

and the neutral kink conjugate $Z^n_{s,\varepsilon}$ is defined via (D3). (The symbol $E_\delta{}^{-1}$ now denotes the operator that vanishes on the one-dimensional kernel of E_δ and equals the inverse of E_δ on its orthocomplement, cf. [2, 17].) Next, we note $\hat{\Gamma}(U(a,R))$ is well defined on account of (3.14); in fact, one has $\hat{\Gamma}(U(a,R)) = \Gamma(U(a,R)_{++})$ due to (3.12). Thus we may introduce the forms

$$\psi^*_{s,\varepsilon}(a,R) \equiv \hat{\Gamma}(U(a,R)) \hat{\Gamma}_r(K^n_{s,\varepsilon}) \Gamma(-1) \hat{\Gamma}(U(a,R))^*, \quad a \in \mathbb{R}^{2N-1}, \quad R \in SO(2N-1) \tag{3.58}$$

and their adjoints, with form domain $\mathscr{D}_{at}$. We also put

$$\psi_{+,j}(a,R) \equiv \begin{pmatrix} \bar{u}_R \\ 0 \end{pmatrix} \cdot \psi_j(0, Ra) \quad j = 1,2, \tag{3.59}$$

$$\psi_{-,j}(a,R) \equiv \begin{pmatrix} 0 \\ \bar{u}_R \end{pmatrix} \cdot \psi_j(0, Ra) \quad j = 1,2, \tag{3.60}$$

cf. (A35). Thus, e.g.,

$$\psi_{+,1}(f,R) \equiv \int da \bar{f}(Ra) \psi_{+,1}(a,R) = B\left(W^{-1} \begin{bmatrix} f u_R \\ 0 \\ 0 \\ 0 \end{bmatrix} \right), \quad f \in L^2(\mathbb{R}^{2N-1}). \tag{3.61}$$

From Lemma A3 it then follows that the fields

$$\{\psi_{s,j}(f,R)\,|\,f\in L^2(\mathbb{R}^{2N-1}),\quad R\in SO(2N-1)\}\tag{3.62}$$

act irreducibly in $\mathscr{F}_a(\mathscr{H}_+)$. The next theorem shows that the forms $\psi_{s,\varepsilon}^{(*)}(a,R)$ can be used to reach this irreducible set of fields for $\varepsilon\to 0$.

Theorem 3.2. *For $N\equiv 1 \bmod 4$ one has*

$$\lim_{\varepsilon\to 0}\psi_{s,\varepsilon}^{(*)}(a,R)=2^{-1/2}(\psi_{s,1}(a,R)-i\psi_{s,2}(a,R))^{(*)}\tag{3.63}$$

in form sense on $\mathscr{D}_{at,0}^{\infty}$.

Proof. This follows in the same way as in the charged case, cf. the proof of Theorem 3.1. Specifically, from Lemma D1 one readily concludes

$$s\text{-}\lim_{\varepsilon\to 0} Z_{s,\varepsilon}^n=-1\tag{3.64}$$

so that, e.g.,

$$\lim_{\varepsilon\to 0}\psi_{+,\varepsilon}^*(0,1)=\lim_{\varepsilon\to 0}\int dp[c^*(p)\cdot\rho_{+,\varepsilon}(p)-c(p)\cdot\overline{\rho'_{+,\varepsilon}(p)}].\tag{3.65}$$

Moreover, using (2.22) one gets

$$(\rho_{+,\varepsilon})_+^{i_1,i_2,j}(p)=(2\pi)^{-(2N-1)/2}\exp(-\varepsilon E_p)(\overline{w_+^{i_1}}(p)\otimes e_{i_2})\cdot\begin{pmatrix}u\\0\end{pmatrix}2^{-1/2}i^{j-1},\tag{3.66}$$

and using (2.20) and (3.18) one gets

$$(\rho'_{+,\varepsilon})_+^{i_1,i_2,j}(p)=-(2\pi)^{-(2N-1)/2}\exp(-\varepsilon E_p)(w_-^{i_1}(p)\otimes e_{i_2})\cdot\begin{pmatrix}\bar{u}\\0\end{pmatrix}2^{-1/2}(-i)^{j-1},\tag{3.67}$$

where $i_1, i_2=1,\dots,n$ and $j=1,2$. Thus, one obtains

$$\lim_{\varepsilon\to 0}\psi_{+,\varepsilon}^*(0,1)=2^{-1/2}\psi^*(0,0)\cdot\begin{bmatrix}u\\iu\\0\\0\end{bmatrix}=2^{-1/2}(\psi_{+,1}(0,1)+i\psi_{+,2}(0,1))\tag{3.68}$$

in form sense. □

We remark that the corresponding result for $N=1$ in [2] differs from (3.63) by a factor $2^{-1/2}$, cf. l.c. Eqs. (4.44), (4.71). This can and should be corrected by adding a factor $2^{-1/2}$ to l.c. Eq. (4.75).

We continue with the case $N\equiv 2 \bmod 4$. Then we choose

$$\mathbb{C}^k=\mathbb{C}^n\tag{3.69}$$

as in the charged case. To satisfy (3.48), we now take as neutral kinks the multipliers

$$\check{K}_{+,\varepsilon}^n\equiv\begin{bmatrix}1_n\otimes\dfrac{\sigma\cdot x+i\varepsilon}{\sigma\cdot x-i\varepsilon} & 0\\ 0 & 1_n\otimes\dfrac{\bar{\sigma}\cdot x-i\varepsilon}{\bar{\sigma}\cdot x+i\varepsilon}\end{bmatrix},\quad \check{K}_{-,\varepsilon}^n\equiv\begin{bmatrix}1_n\otimes\dfrac{\bar{\sigma}\cdot x+i\varepsilon}{\bar{\sigma}\cdot x-i\varepsilon} & 0\\ 0 & 1_n\otimes\dfrac{\sigma\cdot x-i\varepsilon}{\sigma\cdot x+i\varepsilon}\end{bmatrix}.\tag{3.70}$$

Then $K^n_{s,\varepsilon}$ commutes with C, as desired. However, for $m > 0$ the determination of the kernels of the diagonal parts is a problem on which the results obtained thus far shed little light. Indeed, if we proceed as in the proof of Theorem 2.1, then we get as the analog of (2.28), $(2.29)_\delta$, taking e.g. $s = +$:

$$g = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad f = \frac{1}{E_p}\begin{pmatrix} \sigma\cdot p \otimes 1_n G_1 + mG_2 \\ -\sigma\cdot p \otimes 1_n G_2 + mG_1 \end{pmatrix}, \tag{3.71}$$

where G_1, G_2 should obey

$$1_n \otimes \sigma\cdot\nabla G_1 = -\frac{\delta\varepsilon}{E_p}(\sigma\cdot p \otimes 1_n G_1 + mG_2), \tag{3.72}$$

$$1_n \otimes \bar{\sigma}\cdot\nabla G_2 = \frac{\delta\varepsilon}{E_p}(-\sigma\cdot p \otimes 1_n G_2 + mG_1). \tag{3.73}$$

For $m > 0$ we do not know any non-trivial L^2-solutions to this system. Therefore, we leave the case $m > 0$ open and take $m = 0$ from now on. Then we get from Lemma B1 and (A47)

$$\delta = - \Rightarrow G_1 = \alpha \exp(-\varepsilon|p|)u, \quad G_2 = 0, \tag{3.74}$$

$$\delta = + \Rightarrow G_1 = 0, \quad G_2 = \alpha \exp(-\varepsilon|p|)1_n \otimes \bar{V}_c u. \tag{3.75}$$

Thus, Ker $\hat{K}^{n*}_{+,\varepsilon++}$ is spanned by $\kappa_{+,\varepsilon,+}$ and, similarly, Ker $\hat{K}^{n*}_{-,\varepsilon++}$ by $\kappa_{-,\varepsilon,+}$. We set

$$\kappa_{s,\varepsilon} \equiv \kappa_{s,\varepsilon,+} \tag{3.76}$$

and note Ker $\hat{K}^n_{s,\varepsilon++}$ is spanned by

$$\kappa'_{s,\varepsilon} \equiv \hat{C}\hat{K}^{n*}_{s,\varepsilon}\kappa_{s,\varepsilon} = \hat{C}\kappa_{s,\varepsilon,-}. \tag{3.77}$$

We can now implement the kink Bogoliubov transformation with the form (3.55), where the change in meaning of the symbols need not be spelled out. Then we can use (3.59), (3.60) with the subscript j omitted to define Majorana fields $\psi_s^{(*)}(a, R)$, and we can use (3.58) to define approximate Majorana fields $\psi_{s,\varepsilon}^{(*)}(a, R)$. We are now prepared for the last result of this subsection, which justifies this terminology.

Theorem 3.3. *For $N \equiv 2 \bmod 4$ and $m = 0$ one has*

$$\lim_{\varepsilon\to 0} \psi_{s,\varepsilon}^{(*)}(a, R) = \psi_s^{(*)}(a, R) \tag{3.78}$$

in form sense on $\mathscr{D}^{\infty}_{at,0}$.

Proof. This follows as before. □

Appendix A. Finite-Dimensional Clifford Algebras and Spinor Groups

As is well known, the Euclidean Clifford algebra in $\mathbb{R}^{2N}$ has an irreducible $2n$-dimensional representation (recall $n \equiv 2^{N-1}$) which is unique up to unitary equivalence. This representation can be constructed on the fermion Fock space

$$\mathscr{F}_a(\mathbb{C}^N) = \mathbb{C} \oplus \mathbb{C}^N \oplus \wedge^2\mathbb{C}^N \oplus \cdots \oplus \wedge^N\mathbb{C}^N \tag{A1}$$

by choosing

$$\varepsilon_{2j-2} \equiv c_j^* + c_j, \quad \varepsilon_{2j-1} \equiv i(c_j^* - c_j), \quad j = 1, \ldots, N, \tag{A2}$$

where the $c_j^{(*)}$ are the creation/annihilation operators on $\mathscr{F}_a$. Indeed, using the CAR one verifies

$$\{\varepsilon_j, \varepsilon_k\} = 2\delta_{jk}, \quad \varepsilon_j^* = \varepsilon_j, \quad j, k = 1, \ldots, 2N-1. \tag{A3}$$

We shall use 2×2 matrix notation corresponding to the decomposition of $\mathscr{F}_a$ into its sectors of even and odd particle number,

$$\mathscr{F}_a(\mathbb{C}^N) \cong \mathscr{F}_0 \oplus \mathscr{F}_1. \tag{A4}$$

Thus, the matrix of ε_j is of the form $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. From now on we choose orthonormal bases in $\mathscr{F}_0$ and $\mathscr{F}_1$ and correspondingly identify $\mathscr{F}_0$ and $\mathscr{F}_1$ with $\mathbb{C}^n$. Moreover, we choose the bases such that

$$\varepsilon_0 = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}. \tag{A5}$$

Next, we introduce the self-adjoint matrices

$$\alpha_k \equiv i\varepsilon_k \varepsilon_0, \quad \beta \equiv \varepsilon_0. \tag{A6}$$

Then $\{\beta, \alpha_1, \ldots, \alpha_{2N-1}\}$ also satisfy the Euclidean Clifford algebra in $\mathbb{R}^{2N}$. Hence, the α_k can be written

$$\alpha_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}, \tag{A7}$$

where $\{\sigma_1, \ldots, \sigma_{2N-1}\}$ are self-adjoint matrices satisfying the Euclidean Clifford algebra in $\mathbb{R}^{2N-1}$. The time-independent Dirac operator corresponding to a $2N$-dimensional Minkowski space is now given by

$$-i \sum_{j=1}^{2N-1} \alpha_j \partial_j + \beta m, \quad m \geqq 0. \tag{A8}$$

It arises when one writes the time-dependent Dirac equation

$$(\gamma^\mu \partial_\mu + m 1_{2n})\psi = 0 \tag{A9}$$

in Hamiltonian form. Here one has

$$\gamma^0 \equiv \beta, \quad \gamma^k \equiv \beta \alpha_k, \quad k = 1, \ldots, 2N-1 \tag{A10}$$

so that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad g \equiv \mathrm{diag}(1, -1, \ldots, -1). \tag{A11}$$

We continue by proving two lemmas that are essential for proving Lemma B1 below. The first one is concerned with operators

$$a_j \equiv \tfrac{1}{2}\varepsilon_j \otimes \varepsilon_0 \Gamma(-1_N) + \tfrac{1}{2}\varepsilon_0 \otimes \varepsilon_j, \quad j = 1, \ldots, 2N-1 \tag{A12}$$

on the Hilbert space

$$\mathcal{F} \equiv \mathcal{F}_a(\mathbb{C}^N) \otimes \mathcal{F}_a(\mathbb{C}^N), \tag{A13}$$

where Γ denotes the Fock space product operation. (Thus, $\Gamma(-1_N)$ acts as multiplication by $1/-1$ on $\mathcal{F}_0/\mathcal{F}_1$.) The second lemma may be viewed as a corollary of the first.

Lemma A1. *The operators $a_j^{(*)}$ satisfy the* CAR:

$$\{a_j, a_k\} = 0,\ \{a_j, a_k^*\} = 2\delta_{jk},\quad j,k = 1,\ldots,2N-1. \tag{A14}$$

They leave the subspaces

$$\mathcal{F}_+ \equiv (\mathcal{F}_0 \otimes \mathcal{F}_0) \oplus (\mathcal{F}_1 \otimes \mathcal{F}_1),$$
$$\mathcal{F}_- \equiv (\mathcal{F}_0 \otimes \mathcal{F}_1) \oplus (\mathcal{F}_1 \otimes \mathcal{F}_0) \tag{A15}$$

invariant and act irreducibly there. The vector

$$\tilde{\Omega} \equiv v/\|v\|,\quad v \equiv a_1 a_3 \cdots a_{2N-1}\Omega \otimes \Omega \tag{A16}$$

is well defined and may be viewed as the vacuum in $\mathcal{F}_+$. That is, one has

$$a_j\tilde{\Omega} = 0,\quad j = 1,\ldots,2N-1. \tag{A17}$$

Moreover, the number operator

$$\mathcal{N} \equiv \sum_{j=1}^{2N-1} a_j^* a_j \tag{A18}$$

can be written

$$\mathcal{N} = \tfrac{1}{2}(2N-1)1_{2^n} \otimes 1_{2^n} + \frac{1}{2}\sum_{j=1}^{2N-1} \alpha_j \otimes \alpha_j \Gamma(-1_N). \tag{A19}$$

Proof. Using $\{\varepsilon_j, \Gamma(-1_N)\} = 0$ it is straightforward to verify (A14). Since $\Gamma(-1_N) \otimes \Gamma(-1_N)$ has eigenvalue ± 1 on $\mathcal{F}_\pm$, and since the $a_j^{(*)}$ commute with this operator, they leave $\mathcal{F}_+$ and $\mathcal{F}_-$ invariant. Next, we note

$$\dim \mathcal{F}_+ = \dim \mathcal{F}_- = 2^{2N-1} = \dim \mathcal{F}_a(\mathbb{C}^{2N-1}). \tag{A20}$$

Since the $a_j^{(*)}$ satisfy the CAR over $\mathbb{C}^{2N-1}$, this equality implies an irreducible action in $\mathcal{F}_+$ and $\mathcal{F}_-$.

Consider now the vector $v \in \mathcal{F}_+$. Using (A12) and expanding the product, the first terms of each a_j give rise to a vector of the form $\lambda_N c_1^* \cdots c_N^* \Omega \otimes \Omega$ for N even, or $\lambda_N c_1^* \cdots c_N^* \Omega \otimes c_1^* \Omega$ for N odd, with $\lambda_N \neq 0$, cf. (A2). The remaining terms in the expansion cannot cancel this vector, so that $v \neq 0$. To prove (A17) holds true, we first note this is clear for j odd, since $a_j^2 = 0$. Picking now $j = 2l$, one need only verify $a_{2l}a_{2l+1}\Omega \otimes \Omega = 0$, and using (A12) and (A2) this is easy. Finally, (A19) follows from (A12) and (A6). □

Lemma A2. *There exists a unit vector $u \in \mathbb{C}^n \otimes \mathbb{C}^n$, unique up to a phase, which satisfies*

$$(\sigma_j \otimes 1_n)u = \lambda(1_n \otimes \sigma_j)u,\quad \lambda \in \mathbb{R},\quad j = 1,\ldots,2N-1, \tag{A21}$$

if and only if $\lambda=(-)^{N+1}$. *The matrix*

$$\mathscr{P}\equiv(-)^N\sum_{j=1}^{2N-1}\sigma_j\otimes\sigma_j \tag{A22}$$

satisfies

$$\mathscr{P}u=-(2N-1)u \tag{A23}$$

and has eigenvalues $-(2N-5), -(2N-9),\dots,2N-3$ *on the orthocomplement of* u.

Proof. Setting $i\equiv 0,1$ for N even, odd, the vacuum $\tilde{\Omega}$ is in $\mathscr{F}_i\otimes\mathscr{F}_i$. Hence, the number operator $\mathscr{N}$ has spectrum $\{0,2,\dots,2N-2\}$ on $\mathscr{F}_i\otimes\mathscr{F}_i$. Also, using (A19) and (A7) we conclude

$$\mathscr{N}\restriction\mathscr{F}_i\otimes\mathscr{F}_i\cong\tfrac{1}{2}(2N-1)1_n\otimes 1_n+\tfrac{1}{2}\mathscr{P}. \tag{A24}$$

Thus, $\mathscr{P}$ has spectrum $\{-(2N-1), -(2N-5),\dots,2N-3\}$, the spectral value $-(2N-1)$ corresponding to

$$u\equiv\tilde{\Omega} \tag{A25}$$

and being non-degenerate. (Of course, the identification of $\mathscr{F}_0$ and $\mathscr{F}_1$ with $\mathbb{C}^n$ via the choice of bases made above is understood here.) Using the relation

$$\sum_{j=1}^{2N-1}(\sigma_j\otimes 1_n-\lambda 1_n\otimes\sigma_j)^2=(1+\lambda^2)(2N-1)1_n\otimes 1_n-2\lambda(-)^N\mathscr{P}, \tag{A26}$$

the remaining assertions readily follow. □

Next, let $E\in SO(2N)$. Then there exists a unitary matrix $S(E)$, unique up to phase, such that

$$S(E)\varepsilon_j S(E)^*=\sum_{k=0}^{2N-1}E_{kj}\varepsilon_k. \tag{A27}$$

(Indeed, the matrices at the right-hand side are self-adjoint and satisfy the Clifford algebra.) Requiring $\det S(E)=1$, the phase ambiguity is reduced to ± 1 and a faithful representation of the simply-connected spinor group Spin$(2N)$ arises (for $N>1$). Its Lie algebra is spanned by the matrices

$$\varepsilon_{jk}\equiv\varepsilon_j\varepsilon_k\quad 0\leqq j<k\leqq 2N-1 \tag{A28}$$

so that $\mathscr{F}_0$ and $\mathscr{F}_1$ are left invariant. The irreducible representations obtained by restriction are usually denoted Δ_+ and Δ_-.

To avoid confusion, it should be mentioned at this point that the operators $B(f)$ and $\hat{\Gamma}(U)$ of Subsect. 3.3 may be viewed as generalizations of the operators $\varepsilon_0,\dots,\varepsilon_{2N-1}$ and $S(E)$ to an infinite-dimensional context. However, normal ordering is not necessary in the finite-dimensional case, so that ε_{jk} has non-zero vacuum expectation value for $j=2l$, $k=2l+1$, cf. (A2), in contrast to operators of the form $d\hat{\Gamma}(\cdot)$.

It is easily seen that the chiral parts of the standard kinks and hedge-hogs of Subsect. 2.2 and 2.4 belong to $\Delta_\pm$ for fixed $x\in\mathbb{R}^{2N-1}$, but we have no occasion to make use of this. In fact, we will only employ rotations in $SO(2N-1)$, obtained by taking $E_{0j}=E_{j0}=\delta_{0j}$ in (A27). Then one gets

$$[S(R),\beta]=0, \quad S(R)\alpha_j S(R)^* = \sum_{k=1}^{2N-1} R_{kj}\alpha_k, \quad \forall R\in SO(2N-1). \tag{A29}$$

Also, the Lie algebra is spanned by

$$\Sigma_{jk}=\alpha_j\alpha_k=\begin{pmatrix}\sigma_j\sigma_k & 0\\ 0 & \sigma_j\sigma_k\end{pmatrix}, \quad 1\leqq j<k\leqq 2N-1. \tag{A30}$$

Thus, the restrictions of $S(R)$ to $\mathscr{F}_0$ and $\mathscr{F}_1$ are given by two identical $n\times n$ matrices, which will also be denoted $S(R)$. The corresponding irreducible representation of $\mathrm{Spin}(2N-1)$ will be denoted Δ.

From Lemma A1 one can obtain the decomposition of $\Delta\otimes\Delta$ into its irreducible components. Indeed, due to (A12) and (A27) we have

$$S(R)\otimes S(R)a^*(v)S(R)^*\otimes S(R)^* = a^*(Rv), \quad \forall v\in\mathbb{C}^{2N-1}, \quad \forall R\in SO(2N-1), \tag{A31}$$

and since the $a^{(*)}$ satisfy the CAR, one infers

$$S(R)\otimes S(R)\cong\Gamma(R)\oplus\Gamma(R). \tag{A32}$$

(Here we are thinking of $\mathscr{F}$ as $\mathscr{F}_+\oplus\mathscr{F}_-$, cf. Lemma A1.) Therefore, denoting the defining representation of $SO(2N-1)$ by D and noting $\wedge^k D\cong\wedge^{2N-1-k}D$, it follows that

$$\Delta\otimes\Delta\cong\bigoplus_{k=0}^{N-1}\wedge^k D. \tag{A33}$$

(This also follows from the theory of weights, cf. e.g. [19].) Since u spans the vacuum sector in $\mathscr{F}_+$, we get in particular

$$S(R)\otimes S(R)u=u, \quad \forall R\in SO(2N-1). \tag{A34}$$

In Subsects. 3.2 and 3.3 we shall use the following cyclicity result.

Lemma A3. *$\mathbb{C}^n\otimes\mathbb{C}^n$ is spanned by the vectors*

$$u_R\equiv S(R)\otimes 1_n u, \quad R\in SO(2N-1). \tag{A35}$$

Proof. Denote the span of the u_R by V. From (A34) it follows that V can also be written

$$V=\mathrm{span}\{1_n\otimes S(R)u\,|\,R\in SO(2N-1)\}. \tag{A36}$$

Thus, the W^*-algebras

$$\mathscr{A}_L\equiv\{S(R)\otimes 1_n\}'', \quad \mathscr{A}_R\equiv\{1_n\otimes S(R)\}'', \; R\in SO(2N-1) \tag{A37}$$

leave V invariant. But the $S(R)$ act irreducibly on $\mathbb{C}^n$, so that

$$\mathscr{A}_L\equiv\mathscr{L}(\mathbb{C}^n)\otimes 1_n, \quad \mathscr{A}_R=1_n\otimes\mathscr{L}(\mathbb{C}^n). \tag{A38}$$

Hence, V must equal $\mathbb{C}^n\otimes\mathbb{C}^n$. □

Our last topic in this appendix is the charge conjugation matrix U_c, which plays an important role in Sect. 3. Its properties depend on $N \bmod 4$ in a way which can be read off from an explicit representation on $\otimes^N \mathbb{C}^2$ in terms of the Pauli matrices

$$\tau_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A39}$$

Instead of presenting unwieldy general formulas, we shall detail the cases $N = 1, 2, 3, 4$, from which it will be obvious how to continue. Using shorthand illustrated by $\tau_2 \otimes 1_2 \otimes \tau_3 \rightarrow 203$ we set:

$$\begin{array}{llll} N=1: & \varepsilon_0 = 1 & N=2: & \varepsilon_0 = 10 \\ & \varepsilon_1 = 2 & & \varepsilon_1 = 22 \\ & & & \varepsilon_2 = 23 \\ N=4: & \varepsilon_0 = 1000 & & \varepsilon_3 = 21 \\ & \varepsilon_1 = 2222 & & \\ & \varepsilon_2 = 2223 & N=3: & \varepsilon_0 = 100 \\ & \varepsilon_3 = 2221 & & \varepsilon_1 = 222 \\ & \varepsilon_4 = 2230 & & \varepsilon_2 = 223 \\ & \varepsilon_5 = 2210 & & \varepsilon_3 = 221 \\ & \varepsilon_6 = 2300 & & \varepsilon_4 = 230 \\ & \varepsilon_7 = 2100 & & \varepsilon_5 = 210 \end{array} \tag{A40}$$

From this one concludes: (i) (A3) holds true, (ii) one has

$$\varepsilon \equiv \varepsilon_0 \varepsilon_1 \cdots \varepsilon_{2N-1} = i^N \tau_3 \otimes^{N-1} \mathbb{1}_2, \tag{A41}$$

(iii) one has

$$\Omega \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes^{N-1} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \mathscr{F}_0 \cong \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes^{N-1} \mathbb{C}^2, \quad \mathscr{F}_1 \cong \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes^{N-1} \mathbb{C}^2 \tag{A42}$$

(due to (A2) and $\varepsilon \sim \Gamma(-\mathbb{1}_N)$); (iv) the σ_k are obtained from the ε_k by omitting the first entry (since $i\tau_2\tau_1 = \tau_3$, cf. also (A5)–(A7)).

The matrix U_c is the unitary matrix, unique up to phase, such that

$$U_c \bar{\beta} = -\beta U_c, \quad U_c \bar{\alpha}_k = \alpha_k U_c. \tag{A43}$$

Equivalently, U_c satisfies

$$U_c \bar{\varepsilon}_0 = -\varepsilon_0 U_c, \quad U_c \bar{\varepsilon}_k = \varepsilon_k U_c. \tag{A44}$$

From (A40) we then see that we may take

$$U_c = 3, 22, 302, 2202, \ldots \quad N = 1, 2, 3, 4, \ldots. \tag{A45}$$

Hence, we conclude

$$N \equiv 1 \bmod 4: \quad U_c = \begin{pmatrix} V_c & 0 \\ 0 & -V_c \end{pmatrix}, \quad U_c \bar{U}_c = 1_{2n}, \tag{A46}$$

$$N \equiv 2 \bmod 4: \quad U_c = \begin{pmatrix} 0 & V_c \\ -V_c & 0 \end{pmatrix}, \quad U_c \bar{U}_c = 1_{2n}, \tag{A47}$$

$$N \equiv 3 \bmod 4: \quad U_c = \begin{pmatrix} V_c & 0 \\ 0 & -V_c \end{pmatrix}, \quad U_c \bar{U}_c = -1_{2n}, \tag{A48}$$

$$N \equiv 0 \bmod 4: \quad U_c = \begin{pmatrix} 0 & V_c \\ -V_c & 0 \end{pmatrix}, \quad U_c \bar{U}_c = -1_{2n}. \tag{A49}$$

Note that the relation $U_c \bar{U}_c = \pm 1_{2n}$ is base-independent. It is known that for $N \equiv 1 \bmod 4$ one can choose orthonormal bases in $\mathscr{F}_0$ and $\mathscr{F}_1$ such that V_c transforms into 1_n, but we shall not need this. (In view of (A40) this amounts to the existence of a unitary matrix M satisfying $MM^T = 0202$.) We do need the relation

$$U_c \overline{S(R)} = S(R) U_c, \quad \forall R \in SO(2N-1), \tag{A50}$$

which follows by using (A30) and $U_c \overline{\alpha_j \alpha_k} = \alpha_j \alpha_k U_c$, cf. (A43).

Appendix B. A Zero-Mode Lemma

The following lemma is needed to complete the proof of Theorem 2.1.

Lemma B1. *Let $G \in L^2(\mathbb{R}^{2N-1}, dp) \otimes \mathbb{C}^n \otimes \mathbb{C}^n$. Then G satisfies*

$$(1_n \otimes \sigma \cdot \nabla) G = (-)^{N-1} \frac{\delta \varepsilon}{E_p} (\sigma \cdot p \otimes 1_n) G \tag{B1}$$

if and only if

$$\begin{aligned} &G = 0, \quad \delta = +, \\ &G = \alpha \exp(-\varepsilon E_p) u, \quad \alpha \in \mathbb{C}, \quad \delta = -, \end{aligned} \tag{B2}$$

where $u \in \mathbb{C}^n \otimes \mathbb{C}^n$ is the unit vector of Lemma A2.

Proof. This is obvious for $N = 1$, so we take $N > 1$ from now on. Also, from Lemma A2 it follows that (B2) implies (B1). Thus we henceforth assume (B1) holds, and show that this entails (B2). To this end we begin by noting that the multiplier $(\sigma \cdot p \otimes 1_n)/E_p$ is bounded, so that (B1) implies the components of G belong to the Sobolev space $H_1(\mathbb{R}^{2N-1})$. For $m > 0$ the multiplier is smooth. Hence, multiplying (B1) by $1_n \otimes \sigma \cdot \nabla$ it follows that G satisfies the PDE system

$$(-\Delta + W_\delta(p)) G = 0, \tag{B3}$$

$$W_\delta(p) \equiv \frac{\varepsilon^2 p^2}{E_p^2} 1_n \otimes 1_n + (-)^{N-1} \frac{\delta \varepsilon}{E_p} \left(\sum_{j=1}^{2N-1} \sigma_j \otimes \sigma_j - \frac{1}{E_p^2} \sigma \cdot p \otimes \sigma \cdot p \right). \tag{B4}$$

Since W_δ is bounded for $m > 0$, this implies

$$G \in H_2(\mathbb{R}^{2N-1}) \otimes \mathbb{C}^n \otimes \mathbb{C}^n \equiv \mathscr{D}. \tag{B5}$$

Thus, for $m > 0$ the existence of L^2-solutions to (B1) reduces to the existence of zero-energy bound states for the operator

$$H_\delta \equiv -\Delta + W_\delta, \tag{B6}$$

which is clearly self-adjoint on $\mathscr{D}$.

For $m=0$ the multiplier $(\sigma\cdot p\otimes 1_n)/E_p$ is not continuous at 0. However, (B3) still follows for $p\neq 0$. To handle the singularity at $p=0$, we exploit the "uncertainty principle lemma,"

$$\left(\frac{k}{2}-1\right)^2 \int_{\mathbb{R}^k} dp|C(p)|^2\frac{1}{|p|^2}\leqq \int_{\mathbb{R}^k} dp\sum_{j=1}^{k}|(\partial_j C)(p)|^2 \quad C\in H_1(\mathbb{R}^k),\quad k\geqq 3, \tag{B7}$$

(For $C\in C_0^\infty(\mathbb{R}^k)$ this follows by generalizing the $k=3$ argument on p. 169 of [6] in the obvious way; for $C\in H_1$ it then follows by taking limits.) It implies that the components of G are in the domain of multiplication by $1/|p|$. Moreover, since $W_\delta(p)$ is smooth for $p\neq 0$ and Δ is hypo-elliptic, (B3) implies that the components are C^∞ on $\mathbb{R}^{2N-1}\backslash\{0\}$. Using this and the inequality (B7) one readily verifies

$$[1_n\otimes\sigma\cdot\nabla, \sigma\cdot p\otimes 1_n/|p|]G(p)=\frac{1}{|p|}\left(\sum_{j=1}^{2N-1}\sigma_j\otimes\sigma_j-\frac{1}{|p|^2}\sigma\cdot p\otimes\sigma\cdot p\right)G(p), \tag{B8}$$

which holds in the sense that the distributional action on G of the commutator yields the L^2-function at the right-hand side. Thus, (B3) holds weakly for any p, and hence one again obtains $G\in\mathscr{D}$.

Next, we claim that H_δ is self-adjoint on $\mathscr{D}$ for $m=0$, too. To prove this, it suffices to show that the operator of multiplication by

$$\phi(p)/|p|,\quad \phi(r)=\begin{cases}1 & r\leqq R\\ 0 & r>R\end{cases} \tag{B9}$$

(say) is a relatively compact perturbation of Δ, viewed as a self-adjoint operator on $H_2(\mathbb{R}^{2N-1})\subset L^2(\mathbb{R}^{2N-1})$. But this follows by noting that the functions $\phi(|p|)/|p|$ and $1/(p^2+i)$ belong to $L^{2N-2}(\mathbb{R}^{2N-1})$ and using Theorem XI.20 of [7].

The upshot of the above is, that both for $m>0$ and for $m=0$ we are reduced to finding the zero-energy bound states of the self-adjoint operator H_δ with domain $\mathscr{D}$. To this end we note that Lemma A2 entails

$$-(2N-3)\leqq\sum_{j=1}^{2N-1}\sigma_j\otimes\sigma_j\restriction u^\perp\leqq 2N-3, \tag{B10}$$

$$W_\delta(p)u=V_\delta(|p|)u, \tag{B11}$$

where

$$V_\delta(r)\equiv\varepsilon^2r^2/E^2+\delta\varepsilon(2N-1-r^2/E^2)/E,\quad E\equiv(r^2+m^2)^{1/2}. \tag{B12}$$

Combining this with the obvious estimate

$$-1\leqq\frac{1}{E_p^2}\sigma\cdot p\otimes\sigma\cdot p\leqq 1 \tag{B13}$$

we conclude

$$W_\delta(p)\geqq V_-(|p|). \tag{B14}$$

Hence we obtain

$$H_\delta\geqq H\otimes 1_n\otimes 1_n, \tag{B15}$$

where H is the self-adjoint Schrödinger operator

$$H \equiv -\Delta + V_-, \quad \mathscr{D}(H) \equiv H_2(\mathbb{R}^{2N-1}). \tag{B16}$$

We claim that H is a positive operator which has an isolated eigenvalue zero, the corresponding eigenspace being spanned by the function $\exp(-\varepsilon E_p)$. Accepting this for the moment, we can now prove (B2), as follows. First, recall that we have already shown that (B1) entails $G \in \operatorname{Ker} H_\delta$. By virtue of (B15) this implies

$$G = \exp(-\varepsilon E_p)v, \quad v \in \mathbb{C}^n \otimes \mathbb{C}^n. \tag{B17}$$

Since G satisfies (B1), we conclude

$$(\sigma_j \otimes 1_n)v = (-)^N \delta(1_n \otimes \sigma_j)v, \quad j = 1, \ldots, 2N-1, \tag{B18}$$

so invoking Lemma A2 once more we infer $v = 0$ for $\delta = +$ and $v = \alpha u, \alpha \in \mathbb{C}$, for $\delta = -$, which is (B2).

It remains to prove the claim just made. For the special case

$$N = 2, \quad m = 0 \Rightarrow H = -\Delta_3 + \varepsilon^2 - 2\varepsilon/|p|, \tag{B19}$$

this is obvious (at least to a physicist), since $H - \varepsilon^2$ is just the hydrogen atom Hamiltonian. More generally, it is clear that $\exp(-\varepsilon E_p)$ is a zero-energy bound state of H for any $N \geqq 2$ and $m \geqq 0$, and this fact combined with the positivity of $\exp(-\varepsilon E_p)$ will suffice for an expert in Schrödinger operator theory.

We shall, however, add a few details so as to render the proof somewhat more self-contained, and also because in the case at hand we have extra information, compared to the general set-up to be found in Chap. XIII.12 of [8]. First, relative compactness arguments as used in the paragraph containing (B9) imply that H has essential spectrum $[\varepsilon^2, \infty)$. Thus the eigenvalue 0 is isolated, and we have

$$E_0 \equiv \inf \sigma(H) \leqq 0. \tag{B20}$$

Secondly, we note that $\exp(-H)$ is positivity preserving. Indeed, for $m > 0$ this follows from the Trotter product formula for $\exp(-H)$, using the fact that $\exp(\Delta)$ is positivity preserving and V_- is bounded for $m > 0$. Also, using dominated convergence we have

$$s\text{-}\lim_{m \downarrow 0} H(m)\psi = H(0)\psi, \quad \forall \psi \in H_2(\mathbb{R}^{2N-1}) \tag{B21}$$

so that

$$s\text{-}\lim_{m \downarrow 0} \exp(-H(m)) = \exp(-H(0)). \tag{B22}$$

Hence, $\exp(-H)$ is positivity preserving for $m = 0$, too.

Thirdly, suppose ψ is an eigenvector of H with eigenvalue E_0. We may assume ψ is real-valued. Since

$$A \equiv e^{-H} \leqq e^{-E_0} \tag{B23}$$

is positivity preserving, we have

$$0 \leqq (|\psi| - \psi, A(|\psi| + \psi)) = (|\psi|, A|\psi|) - (\psi, A\psi). \tag{B24}$$

Hence,

$$(|\psi|, A|\psi|) \geqq (\psi, A\psi) = e^{-E_0}(|\psi|, |\psi|). \tag{B25}$$

In view of (B23) this implies $|\psi|$ is an eigenvector of H with eigenvalue E_0, too. Thus, we must have $E_0 = 0$ and $|\psi| \sim \exp(-\varepsilon E_p)$, since $|\psi|$ cannot be orthogonal to the positive function $\exp(-\varepsilon E_p)$. Moreover, since ψ satisfies the PDE $H\psi = 0$, it must be continuous for $p \neq 0$. Because $|\psi|$ does not vanish, we must have $\psi = |\psi|$ or $\psi = -|\psi|$. Thus, 0 is a simple eigenvalue of H and the proof is complete. □

For $m > 0$ the Dirac operator involved in (B1) satisfies the assumptions guaranteeing that the index theorems of Callias [20] and Hörmander [21] apply, cf. also [22]. Consequently, its index can be written in terms of an integral over S^{2N-2}. Since the index can be read off from Lemma B1, the value of the integral follows as a corollary. Conversely, if one is able to calculate the integral, then the value of the index results. This would suffice for the index formulas of Subsect. 2.4. However, in Sect. 3 the far more explicit information of Lemma B1 is indispensable.

Appendix C. Compactness and Non-Compactness

Due to Schwartz's nuclear theorem any bounded operator K on $L^2(\mathbb{R}^l, dp)$ can be represented by a tempered distribution $K(p,q) \in S'(\mathbb{R}^{2l})$. In this appendix we isolate conditions on $K(p,q)$ guaranteeing compactness or non-compactness of K (Lemmas C1 and C2). We also prove a lemma (Lemma C3) that will enable us to show that certain operators occurring in the main text are not Hilbert–Schmidt.

Lemma C1. *Let K be an operator on $L^2(\mathbb{R}^l)$ with kernel*

$$K(p,q) = F(p-q)B(p,q), \quad F \in L^1(\mathbb{R}^l),\ B \in L^\infty(\mathbb{R}^{2l}). \tag{C1}$$

Assume that for any $r > 0$ one has

$$\lim_{R\to\infty} \sup_{\substack{|x|>R \\ |y|<r}} |B(x+y, x)| = 0, \tag{C2}$$

$$\lim_{R\to\infty} \sup_{\substack{|x|>R \\ |y|<r}} |B(x, x+y)| = 0. \tag{C3}$$

Then K is compact.

Proof. The proof is based on two well-known facts. First, norm limits of compact operators are compact, and second, an operator T with measurable kernel $T(p,q)$ satisfies

$$\|T\|^2 \leqq \sup_q \int dp\, |T(p,q)| \sup_p \int dq\, |T(p,q)|. \tag{C4}$$

(This follows either from the Riesz–Thorin theorem or directly from a slightly subtle application of the Schwarz inequality.)

Due to (C4) and (C1) we have

$$\|K\| \leqq \|B\|_\infty \|F\|_1. \tag{C5}$$

Now C_0^∞ is dense in L^1, so we need only prove compactness of K for $F \in C_0^\infty$ by virtue of (C5) and the first fact. Thus we assume from now on

$$\sup F \subset B_r \equiv \{y \in \mathbb{R}^l \mid |y| < r\}. \tag{C6}$$

Next, we take $R > 2r$ and set

$$K \equiv K_R + H_R, \quad K_R(p,q) \equiv [1 - \chi_R(p)\chi_R(q)]K(p,q), \tag{C7}$$

where χ_R denotes the characteristic function of B_R. Then $H_R(p,q)$ has support in $B_R \times B_R$, and since F and B are bounded, one concludes $H_R(p,q) \in L^2(\mathbb{R}^{2l})$. Thus, H_R is Hilbert–Schmidt. Invoking the first fact once more, it remains to prove

$$\lim_{R \to \infty} \|K_R\| = 0. \tag{C8}$$

To this end we exploit the second fact, cf. (C4). We shall show

$$\sup_q \int dp |K_R(p,q)| \to 0, \quad R \to \infty \tag{C9}$$

by invoking (C2); the second supremum behaves in the same way due to (C3). (In fact, the alert reader will have noted that (C2) and (C3) are equivalent.) To prove (C9) we fix q and consider

$$\int dp |K_R(p,q)| = \int_{B_r} dy [1 - \chi_R(q+y)\chi_R(q)] |F(y)B(q+y,q)|, \tag{C10}$$

cf. (C6)–(C7). Since $r < R/2$, the function in brackets vanishes on B_r when $|q| \leqq R/2$. Thus, we obtain

$$\int dp |K_R(p,q)| \leqq \sup_{|x| > R/2} \int_{|y| < r} dy |F(y)B(x+y,x)| \tag{C11}$$

for any $q \in \mathbb{R}^l$. As promised, this yields (C9) due to (C2). □

Lemma C2. *Let K be an operator on $L^2(\mathbb{R}^l)$ whose kernel $K(p,q) \in S'(\mathbb{R}^{2l})$ has the following property: There exist $f, g \in S(\mathbb{R}^l)$ and $e \in S^{l-1}$ such that*

$$\lim_{\lambda \to \infty} \int dp\, dq \bar{f}(p) K(p + \lambda e, q + \lambda e) g(q) \neq 0. \tag{C12}$$

(Here, the integral stands for distributional evaluation.) Then K is not compact.

Proof. We assume K is compact and derive a contradiction. Denote by U_λ the translation over λe. Then U_λ weakly converges to 0 for $\lambda \to \infty$ by the Riemann–Lebesgue lemma. Hence, KU_λ converges strongly to 0, so that

$$s\text{-}\lim_{\lambda \to \infty} U_{-\lambda} K U_\lambda = 0. \tag{C13}$$

But this implies

$$\lim_{\lambda \to \infty} (U_\lambda f, K U_\lambda g) = 0, \tag{C14}$$

which contradicts (C12). □

Lemma C3. *Let $f \geqq 0$ be a measurable function on $\mathbb{R}^l$, $l > 1$, and let $c \geqq 0$. Then one has*

$$\int_{\mathbb{R}^{2l}} dx\,dy f(y)\left(\frac{(x+y)^2}{(x+y)^2+c}+\frac{(x-y)^2}{(x-y)^2+c}-2\frac{(x^2-y^2)}{[((x+y)^2+c)((x+y)^2+c)]^{1/2}}\right)<\infty \tag{C15}$$

if and only if $f=0$.

Proof. We assume $f\neq 0$ and show that when (C15) holds a contradiction arises. Indeed, (C15) implies by virtue of Fubini's theorem that there exists $y_0\neq 0$ with $f(y_0)>0$ such that the x-integral of the bracketed function with $y=y_0$ converges. Now introduce

$$x\equiv re,\quad e\in S^{l-1},\quad a\equiv y_0^2,\quad b\equiv e\cdot y_0. \tag{C16}$$

Invoking Fubini's theorem again, we infer that there exists e such that $a\neq b^2$ and

$$\int_0^\infty dr\,r^{l-1}I(r)<\infty, \tag{C17}$$

where

$$I(r)\equiv\frac{r^2+2br+a}{r^2+2br+a+c}+\frac{r^2-2br+a}{r^2-2br+a+c}-2\frac{(r^2-a)}{[(r^2+2br+a+c)(r^2-2br+a+c)]^{1/2}}. \tag{C18}$$

But one has

$$\begin{aligned}I(r)&=2\left(1-\frac{c}{r^2}\right)-2\left(1-\frac{a}{r^2}\right)\left(1+\frac{1}{r^2}[2b^2-a-c]\right)+O(r^{-3})\\&=\frac{4}{r^2}(a-b^2)+O(r^{-3}),\quad r\to\infty\end{aligned} \tag{C19}$$

and since $a\neq b^2$ and $l>1$, this contradicts (C17). □

Appendix D. A Convergence Lemma

This appendix contains the definition of the kink conjugates $Z_{s,\varepsilon}$ and a lemma that is a crucial ingredient in the proofs of Theorems 3.1–3.3. We shall use notation explained in Subsect. 3.1. Suppose U is a unitary with compact off-diagonal parts for which $\mathrm{Ker}\,U^*_{--}$ is trivial and $\mathrm{Ker}\,U_{--}$ is one-dimensional. Then $\mathrm{Ker}\,U_{++}$ is trivial and $\mathrm{Ker}\,U^*_{++}$ is one-dimensional, since U is unitary. Thus, the operators

$$E_-\equiv U_{--}U^*_{--}=P_--U_{-+}U^*_{+-}, \tag{D1}$$

$$E_+\equiv U^*_{++}U_{++}=P_+-U^*_{+-}U_{-+} \tag{D2}$$

have bounded inverses (as operators on H_δ), and we can define an operator Z by

$$\begin{aligned}Z_{++}&\equiv -U_{++}E_+^{-1}, & Z_{+-}&\equiv -U_{++}E_+^{-1}U_{-+}^*,\\ Z_{-+}&\equiv -U^*_{--}E_-^{-1}U_{-+}, & Z_{--}&\equiv -U^*_{--}E_-^{-1}.\end{aligned} \tag{D3}$$

This operator is referred to as the conjugate of U; it is related to the associate Λ used in [16] by $Z(U)=1+(P_+-P_-)\Lambda(-U)$. (This sign convention ensures absence of signs in the associated Fock space implementer, cf. Eq. (5.15) in [16].)

We have shown that the operators $K_{s,\varepsilon}$ satisfy all of the above assumptions, cf. Theorems 2.1 and 2.5. Thus we may introduce operators $E_{\pm,s,\varepsilon}$ and $Z_{s,\varepsilon}$ in the way just described. This prepares us for the following result.

Lemma D1. *One has*

$$s\text{-}\lim_{\varepsilon\to 0} Z_{s,\varepsilon} = -1, \quad s=+,-. \tag{D4}$$

Proof. Once this is proved for $s=+$, the $s=-$ case follows by using parity. Thus we take $s=+$ and suppress this index henceforth. First, we shall handle the massless case. We claim that

$$\|K_{\varepsilon-+}\| = C < 1, \quad \forall \varepsilon > 0, \quad m=0. \tag{D5}$$

Indeed, using (2.13) we obtain

$$\|K_{\varepsilon-+}\| = \|D(\varepsilon)^* P_- K_1 P_+ D(\varepsilon)\| = \|K_{1-+}\| \equiv C, \tag{D6}$$

and since $E_{+,1}$ has a bounded inverse we must have $C<1$, cf. (D2).

Next, we note

$$s\text{-}\lim_{\varepsilon\to 0} K_\varepsilon = 1. \tag{D7}$$

(This is immediate from (2.16).) By majorizing the tail in the Neumann series for $E_\delta{}^{-1}$ with the uniform bound (D5) it readily follows from this that

$$s\text{-}\lim_{\varepsilon\to 0} E_{\delta,\varepsilon}{}^{-1} = P_\delta. \tag{D8}$$

Moreover, combining (D1), (D2) with (D5) we get

$$\|E_{\delta,\varepsilon}{}^{-1}\| = (1-C^2)^{-1/2}, \quad \forall \varepsilon > 0, \quad m=0. \tag{D9}$$

Using all this, it is routine to verify (D4): One has, e.g.,

$$\|(Z_{\varepsilon++} + P_+)f\| \leqq \|K_{\varepsilon++} E_{+,\varepsilon}{}^{-1}(E_{+,\varepsilon} - P_+)f\| + \|(K_{\varepsilon++} - P_+)f\| \to 0, \quad \varepsilon \to 0, \tag{D10}$$

cf. (D3).

The massive case involves more work. Suppose we can show

$$\|K_{\varepsilon-+}\| \leqq C' < 1, \quad \forall \varepsilon \in (0,1], \quad m>0. \tag{D11}$$

Then we can argue as in the massless case to prove (D4); we need only replace (D9) by

$$\|E_{\delta,\varepsilon}{}^{-1}\| \leqq (1-C'^2)^{-1/2}, \quad \forall \varepsilon \in (0,1], \quad m>0. \tag{D12}$$

To prove (D11) we observe that

$$\|K_{\varepsilon-+}\| = \|\hat{K}_{\varepsilon-+}\| = \|\hat{P}_{(m)-} \hat{D}(\varepsilon)^* \hat{K}_1 \hat{D}(\varepsilon) \hat{P}_{(m)+}\| = \|\hat{P}_{(\varepsilon m)-} \hat{K}_1 \hat{P}_{(\varepsilon m)+}\| \equiv f(\varepsilon m). \tag{D13}$$

(Here and from now on the mass dependence of $\hat{P}_\delta$ is made explicit.) Since $\hat{P}_{(\mu)\delta}$ is norm continuous in μ on $(0,\infty)$, the function $f(\mu)$ is continuous on $(0,1]$.

Moreover, one has $f(\mu)<1$ on $(0,1]$. Therefore, f is bounded away from 1 on compact subintervals of $(0,1]$. But then we need only show

$$\lim_{\mu\to 0} f(\mu)=\|\hat{P}_{(0)-}\hat{K}_1\hat{P}_{(0)+}\|=C<1 \tag{D14}$$

to obtain the desired bound (D11).

We shall prove (D14) by making suitable use of the estimate

$$\|f(p)g(-i\nabla_p)\|_r \leqq (2\pi)^{-l/r}\|f\|_r\|g\|_r, \quad f,g\in L^r(\mathbb{R}^l), \quad r\in[2,\infty), \tag{D15}$$

where $\|\cdot\|_r$ denotes the Schatten norm at the left and the L^r norm at the right, cf. Theorem XI.20 in [7]. To this end we introduce (omitting the carets from now on)

$$C(\mu)\equiv[P_{(\mu)-}-P_{(0)-},K_1] \tag{D16}$$

and note

$$C(\mu)=(P_{(\mu)-}(p)-P_{(0)-}(p))M(-i\nabla_p)-M(-i\nabla_p)(P_{(\mu)-}(p)-P_{(0)-}(p)),$$
$$M(x)\equiv\check{K}_1(x)-1. \tag{D17}$$

Now it is easily seen that the matrix elements of $M(x)$ and of $P_{(\mu)-}(p)-P_{(0)-}(p)$ belong to $L^r(\mathbb{R}^{2N-1})$ for $r\in(2N-1,\infty]$. Moreover, by dominated convergence the latter matrix elements converge to 0 in L^r for $\mu\to 0$ and $r\in(2N-1,\infty)$. Hence, the estimate (D15) entails

$$\lim_{\mu\to 0}\|C(\mu)\|=0, \tag{D18}$$

since the operator norm is dominated by any Schatten norm.

Next, we multiply $C(\mu)$ by $P_{(\mu)-}$ from the left and by $P_{(\mu)+}$ from the right, and conclude

$$\lim_{\mu\to 0}\|P_{(\mu)-}K_1P_{(\mu)+}-P_{(\mu)-}P_{(0)-}K_1P_{(0)+}P_{(\mu)+}+P_{(\mu)-}P_{(0)+}K_1P_{(0)-}P_{(\mu)+}\|=0. \tag{D19}$$

Since the projections $P_{(\mu)\delta}$ strongly converge to $P_{(0)\delta}$ for $\mu\to 0$, the norm of the second operator has limit $\|P_{(0)-}K_1P_{(0)+}\|$. Hence, (D14) will result from (D19), provided one has

$$\lim_{\mu\to 0}\|P_{(\mu)-}P_{(0)+}K_1P_{(0)-}P_{(\mu)+}\|=0. \tag{D20}$$

But this can be proved by another application of (D15): We may replace K_1 by K_1-1, and since $P_{(\mu)-}(p)P_{(0)+}(p)$ has matrix elements that converge to 0 in L^r for $\mu\to 0$ and $r\in(2N-1,\infty)$, (D20) holds true. □

Appendix E. The Connection to External Field S-Operators

In this appendix we present some results on the (interaction picture) evolution operators and S-operators corresponding to the Dirac equation with certain time-dependent external fields. This will yield a different context for the above results, which is closer to the physical picture of chiral anomalies [23]. We shall make use of concepts and results that are detailed in [24, 25]. Using the notation of Subsect. 2.1, the external field Dirac operator is given by

$$H(t) \equiv H + \lambda V(t). \tag{E1}$$

Here, $\lambda \in \mathbb{C}$ is the coupling constant and

$$(\check{V}(t)f)(x) \equiv V(t,x)f(x), \quad f \in \check{\mathscr{H}}, \tag{E2}$$

where $V(t,x)$ is a $2nk \times 2nk$ matrix-valued function on $2N$-dimensional Minkowski space.

First, we shall assume $V(t,x)$ is continuous and vanishes at ∞, so that $\| V(t) \|$ is continuous and vanishes at ∞. In addition, we assume

$$\| V(t) \| \in L^1(\mathbb{R}). \tag{E3}$$

These two assumptions guarantee that the evolution operator $U_\lambda(T_2, T_1)$ is norm entire in λ and norm continuous on $\tilde{\mathbb{R}}^2$, where $\tilde{\mathbb{R}} \equiv \mathbb{R} \cup \{\pm \infty\}$ with the obvious topology, cf. [24], Sect. 2.

Theorem E1. *For any $(\lambda, T_1, T_2) \in \mathbb{C} \times \tilde{\mathbb{R}}^2$ the operator $U_\lambda(T_2, T_1)$ has compact off-diagonal parts and Fredholm diagonal parts with vanishing index.*

Proof. We shall first prove this under the extra assumption that the matrix elements of $V(t,x)$ are in $S(\mathbb{R}^{2N})$. Then the operators $U \equiv U_\lambda(\infty, -\infty)$ and $V \equiv U_\lambda(-\infty, \infty)$ have Hilbert–Schmidt off-diagonal parts, as follows by generalizing the relevant arguments of [25] in a straightforward way, cf. also [26]. Since $UV = VU = 1$, it follows that U and V have Fredholm diagonal parts; furthermore, these have index 0 since U and V are norm entire in λ and equal to 1 for $\lambda = 0$. Now consider (e.g.) $U_\lambda(T, 0)$ with $T \in (0, \infty)$. Multiply $V(t,x)$ by a C^∞ function $\phi_\varepsilon(t)$ that is 1 on $[\varepsilon, T-\varepsilon]$, 0 on $(-\infty, 0]$ and $[T, \infty)$, and monotone on $[0, \varepsilon]$ and $[T-\varepsilon, T]$. Then the corresponding evolution operator $U_{\lambda,\varepsilon}(T, 0)$ equals the S-operator for the Schwartz space external field $\lambda\phi_\varepsilon(t)V(t,x)$ and, therefore, has HS off-diagonal parts. Using the Dyson expansion to estimate $U_\lambda(T,0) - U_{\lambda,\varepsilon}(T,0)$, it readily follows that this difference converges to 0 in norm for $\varepsilon \to 0$. Hence, $U_\lambda(T,0)_{\delta-\delta}$ and, similarly, $U_\lambda(T_2, T_1)_{\delta-\delta}$ are compact.

Next, consider the general case. Since $V(t,x)$ is continuous and vanishes at ∞, one can find a family $V_\varepsilon(t,x)$ with matrix elements in $S(\mathbb{R}^{2N})$ such that

$$\| V_\varepsilon(t) - V(t) \| \leqq \varepsilon, \quad \forall t \in [T_1, T_2] \subset \mathbb{R}. \tag{E4}$$

Telescoping the Dyson expansion in the obvious way, one infers $n\text{-}\lim_{\varepsilon \to 0} U_{\lambda,\varepsilon}(T_2, T_1) = U_\lambda(T_2, T_1)$. Thus the assertions follow for $\lambda \in \mathbb{C}$ and $T_i \in \mathbb{R}$, and taking norm limits for $T_i \in \tilde{\mathbb{R}}$, too. □

The second assumption (E3) is critical. Indeed, in [1] Matsui proves (for $N = 2$ and $m = 0$) there exist external fields that are continuous and vanish at ∞, yet lead to an S-operator with index $S_{--} \neq 0$. For these fields one has $\| V(t) \| \sim |t|^{-1}$ for large times, so (E3) is violated. His fields are in essence pure gauge for large times, but they have a time dependence which leads to considerable complications. Here, we shall obtain S-operators with non-zero index (for $N \geqq 1$ and $m \geqq 0$)

corresponding to external fields that are time-independent for large times (so that both assumptions are violated).

Specifically, let us assume $V(t,x)$ is continuous and self-adjoint on $\mathbb{R}^{2N}$, vanishes for $|x|\to\infty$ and t fixed, and is equal to time-independent matrices $V_\tau(x)$ for $\tau t \geqq T > 0$, where $\tau = +, -$. Then the issue of existence of the S-operator for the field $V(t,x)$ reduces to the existence problem for the wave operators $W_+(H, H_+)$ and $W_-(H_-, H)$, where

$$\check{H}_\tau \equiv \check{H} + \lambda V_\tau(\cdot), \quad \tau = +, -. \tag{E5}$$

Indeed, one has

$$U(t,s) = e^{itH} e^{-i(t-T)H_+} e^{-iTH} U(T, -T) e^{-iTH} e^{i(s+T)H_-} e^{-isH}, \quad t > T, \quad s < -T, \tag{E6}$$

cf. Eqs. (2.27), (2.106) in [24]. If $V_\pm \neq 0$, the norm limits of the right-hand side for $t\to\infty$ and $s\to-\infty$ do not exist, but the strong limits may exist. Since Fredholm indices can jump under strong limits, the S-operators associated with such fields may yield diagonal parts with non-zero index.

To study this, we further restrict ourselves to the case

$$H_\tau \equiv U_\tau^* H U_\tau, \quad \check{U}_\tau \equiv \begin{pmatrix} 1_n \otimes u_{\tau,+}(\cdot) & 0 \\ 0 & 1_n \otimes u_{\tau,-}(\cdot) \end{pmatrix}, \tag{E7}$$

where $u_{\tau,s}(x)$ are $U(k)$-valued functions with the following properties:

$$u_{\tau,s}(x) \in C^1, \quad \tau, s = +, -, \tag{E8}$$

$$\nabla u_{\tau,s}(x) = o(1), \quad |x| \to \infty. \tag{E9}$$

Then it is easy to verify that $\check{H}_\tau$ is indeed of the previously assumed form (E5), with $\lambda \equiv 1$, say. Specifically, one obtains

$$\begin{aligned} V(t,x) = {} & i \sum_{j=1}^{2N-1} \begin{pmatrix} -\sigma_j \otimes u_{\tau,+}^*(x)(\partial_j u_{\tau,+})(x) & 0 \\ 0 & \sigma_j \otimes u_{\tau,-}^*(x)(\partial_j u_{\tau,-})(x) \end{pmatrix} \\ & + m \begin{pmatrix} 0 & 1_n \otimes [u_{\tau,+}^*(x) u_{\tau,-}(x) - 1_k] \\ 1_n \otimes [u_{\tau,-}^*(x) u_{\tau,+}(x) - 1_k] & 0 \end{pmatrix}, \quad \tau t \geqq T. \end{aligned} \tag{E10}$$

Recall that the interpolation of the two fields involved need only be continuous, self-adjoint and 0 at ∞. First, we require in addition to (E8), (E9),

$$u_{\tau,s}(x) - 1_k = o(1), \quad |x| \to \infty. \tag{E11}$$

Thus one has $U_\tau \in G_e \subset G_\infty$, cf. Subsect. 2.4.

Theorem E2. *Under the assumptions just made, the S-operator*

$$\operatorname*{s\cdot lim}_{\substack{T_2\to\infty \\ T_1\to-\infty}} U(T_2, T_1) \tag{E12}$$

exists and is given by

$$e^{iTH} U_+ e^{-iTH} U(T, -T) e^{-iTH} U_-^* e^{iTH} \equiv S. \tag{E13}$$

Moreover, the operators $S_{\pm\mp}$ are compact and the operators $S_{\pm\pm}$ are Fredholm, and one has

$$\operatorname{index} S_{\delta\delta} = \operatorname{index} U_{+\delta\delta} - \operatorname{index} U_{-\delta\delta}. \tag{E14}$$

Proof. In the case at hand we can rewrite (E6) as

$$U(t,s) = [e^{itH} U_+^* e^{-itH}] S [e^{isH} U_- e^{-isH}], \quad t > T, \quad s < -T. \tag{E15}$$

Therefore, the first assertion follows if we show that the bracketed operators have strong limit 1 for $t \to \infty$ and $s \to -\infty$, respectively. To this end we need only prove

$$s\text{-}\lim_{t\to\infty} M(\cdot)\exp(it\check{H}) = 0, \tag{E16}$$

where $M(x)$ is continuous and vanishes at ∞. Let f be of the form $(\check{H}+i)^{-1}g, g\in\check{\mathscr{H}}$. An application of the trace ideal estimate (D15) shows that $M(x)(-i\alpha\cdot\nabla + \beta m + i)^{-1}$ is compact when $M(x) = O(|x|^{-1})$ (say), and hence (taking a norm limit) when $M(x) = o(1)$ for $|x| \to \infty$, too. Since $\exp(it\check{H})$ weakly converges to 0 for $t \to \infty$ by the Riemann–Lebesgue lemma, we conclude

$$M(\cdot)\exp(it\check{H})f = M(\cdot)(\check{H}+i)^{-1}\exp(it\check{H})g \xrightarrow{s} 0, \quad t \to \infty. \tag{E17}$$

From this (E16) readily follows.

Next, we note that Theorem E1 implies $U(T,-T)_{\delta-\delta}$ are compact and $U(T,-T)_{\delta\delta}$ are Fredholm with index 0. Since $U_\tau \in G_\infty$, this entails the validity of the second assertion. □

We shall now relax the assumption (E11). Assume continuous functions $u_{\tau,\infty}: S^{2N-2} \to U(k)$ exist such that

$$u_{\tau,s}(x) - u_{\tau,\infty}\left(\frac{x}{|x|}\right) = o(1), \quad |x| \to \infty, \quad \tau, s = +, -. \tag{E18}$$

That is, we allow hedge-hog asymptotics at ∞. In particular, $U_\tau \in G_h \subset G_\infty$ for $m > 0$, cf. Subsect. 2.4.

Theorem E3. *Under these assumptions the S-operator (E12) exists and is given by*

$$S_h = u_{+,\infty}^*\left(\frac{p}{|p|}\frac{H}{E_p}\right) S u_{-,\infty}\left(-\frac{p}{|p|}\frac{H}{E_p}\right), \tag{E19}$$

where S is defined by (E13). Moreover, for $m > 0$ the operators $S_{h\pm\mp}$ are compact and the operators $S_{h\pm\pm}$ are Fredholm, and one has

$$\operatorname{index} S_{h\delta\delta} = \operatorname{index} U_{+\delta\delta} - \operatorname{index} U_{-\delta\delta}. \tag{E20}$$

Proof. Because (E15) still holds when (E11) is replaced by (E18), and because of the above argument containing (E16), we need only show

$$s\text{-}\lim_{t\to\infty} u_{+,\infty}^*\left(e^{itH}\mathscr{F}\frac{x}{|x|}\mathscr{F}^{-1}e^{-itH}\right) = u_{+,\infty}^*\left(\frac{p}{|p|}\frac{H}{E_p}\right), \tag{E21}$$

$$s\text{-}\lim_{t\to-\infty} u_{-,\infty}\left(e^{itH}\mathscr{F}\frac{x}{|x|}\mathscr{F}^{-1}e^{-itH}\right) = u_{-,\infty}\left(-\frac{p}{|p|}\frac{H}{E_p}\right). \tag{E22}$$

(The notation used here calls for a comment: When $A_j \equiv a_j \otimes 1_k$, $j=1,\ldots,2N-1$, are commuting self-adjoint operators and u a map from $\mathbb{R}^{2N-1}$ to $U(k)$, then the operators $u_{lm}(a_1,\ldots,a_{2N-1})$ are defined by the functional calculus, and

$$u(A) \equiv \sum_{l,m=1}^{k} u_{lm}(a) \otimes e_{lm} \tag{E23}$$

is unitary, where $\{e_{lm}\}$ is the obvious basis of $M_k(\mathbb{C})$.)

To prove (E21), (E22) we set

$$v_j(t) \equiv e^{itH}(i\partial_j/t)e^{-itH}, \quad t \neq 0 \tag{E24}$$

and exploit the fact that

$$\lim_{|t|\to\infty} v_j(t) = p_j/H \quad \text{(strong resolvent sense).} \tag{E25}$$

Taking (E25) for granted, it follows that

$$s\text{-}\lim_{|t|\to\infty} v_j(t)/|v(t)| = p_j H/|p| E_p. \tag{E26}$$

(To see this, note the discontinuity of the function at the left-hand side is harmless, since $v_j(t)$ has no point spectrum.) Since one has

$$e^{itH}\mathscr{F}(x_j/|x|)\mathscr{F}^{-1}e^{-itH} = \pm v_j(t)/|v(t)|, \quad t \gtrless 0, \tag{E27}$$

and since $u^*_{+,\infty}$ and $u_{-,\infty}$ are continuous on S^{2N-2}, (E21) and (E22) follow.

It remains to prove the relation (E25). Its validity was first shown by Thaller and Enss [27], who were studying the (interacting) $N=2$, $m>0$ case, but their argument generalizes to any $N \geqq 1$ and $m \geqq 0$. (Cf. also [1] for what follows.) Indeed, following [27], we set $F_j \equiv \alpha_j - p_j/H$ and note $F_j H = -HF_j$, so that $F_j e^{-itH} = e^{itH}F_j$. This can be rewritten

$$e^{itH}[\partial_j, H]e^{-itH} = \frac{p_j}{H} + e^{2itH}\left(\alpha_j - \frac{p_j}{H}\right). \tag{E28}$$

Denoting the domain of $i\partial_j$ by $\mathscr{D}$, one readily verifies $e^{itH}\mathscr{D} = \mathscr{D}$. Hence, (E28) entails that on $\mathscr{D}$ one has

$$e^{itH}(i\partial_j/t)e^{-itH} = i\partial_j/t + \frac{p_j}{H} + \frac{(e^{2itH}-1)}{2itH}\left(\alpha_j - \frac{p_j}{H}\right). \tag{E29}$$

But it is obvious that the first term converges strongly to 0 on $\mathscr{D}$, whereas the third term has norm $\leqq 2$ and strongly converges to 0 by virtue of a routine argument. Hence, (E25) follows. □

It is clear from the above proofs that the assumptions can be relaxed to obtain similar conclusions, but we shall not pursue this. We do point out that one may allow $V(t,x)$ to have jumps as a function of time. (Indeed, this is clear from the relation $U(T_3,T_1) = U(T_3,T_2)U(T_2,T_1)$.) In particular, we may take $T=0$ and $U_- = 1$ in the above. Then we conclude that unitary multipliers $\check{U} \in G_e$ for which $u_s(x)$, $s=+,-$, satisfy (E8), (E9) may be viewed as S-operators corresponding to external fields that vanish for $t<0$ and are given by the right-hand side of (E10) (with $u_{\tau,s} \to u_s$) for $t \geqq 0$.

In particular, the standard kinks $K_{s,\varepsilon}$ of Subsect. 2.2 and any finite product of their transforms under translations and rotations satisfy (E8), (E9) and (E11). From the viewpoint sketched in this appendix the results of Theorem 2.1 can be interpreted as follows: Scattering at the external field associated with $K_{s,\varepsilon}$ via the right-hand side of (E10) can move states in the negative energy subspace of H to the positive energy subspace (viz., those states proportional to $\kappa_{s,\varepsilon,-}$), but not vice versa. Equivalently, these states are negative energy states with respect to H, but positive energy states with respect to H_+, defined via (E7) with $U_+ \equiv K_{s,\varepsilon}$, but no states exist that have positive energy with respect to H and negative energy with respect to H_+.

For the (massive) standard hedge-hogs (2.93) the assumptions (E8), (E9) and (E18) are fulfilled. Hence, the index of the corresponding S_{h--} equals 1, cf. Theorem E3. However, in this case we have no explicit information on the relevant kernel states but for the dimension difference.

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